Combined Variance Reduction Techniques in Fully Sequential Selection Procedures

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Abstract

In the past several decades, many ranking-and-selection (R&S) procedures have been developed to select the best simulated system with the largest (or smallest) mean performance measure from a finite number of alternatives. A major issue to address in these R&S problems is to balance the trade-off between the effectiveness (i.e., making a correct selection with a high probability) and the efficiency (i.e., using a small total number of observations). In this paper, we take a frequentist's point of view by setting a predetermined probability of correct selection while trying to reduce the total sample size, that is, to improve the efficiency but also maintain the effectiveness. In particular, in order to achieve this goal, we investigate combining various variance reduction techniques into the fully sequential framework, resulting in different R&S procedures with either finite-time or asymptotic statistical validity. Extensive numerical experiments show great improvement in the efficiency of our proposed procedures as compared with several existing procedures.
Keywords: ranking and selection; control variates; conditional expectation; poststratified sampling; simulation; variance reduction techniques

1 Introduction

Selecting the best system, one with the largest (or smallest) performance measure from a finite number of alternatives, is known as a ranking-and-selection (R&S) problem in the simulation literature. The original work on R&S can be traced back at least to Bechhofer [3] that established the indifference-zone (IZ) formulation by assuming the difference between the best and second-best systems is greater than or equal to an IZ parameter. Since then, many procedures have been designed to solve R&S problems, which in general can be classified into two categories: the Bayesian approach and the frequentist approach. Interested readers may refer to Chick [7] and Kim and Nelson [23] for a comprehensive review of both the Bayesian and frequentist methods.

For many existing R&S procedures in the literature, a key objective is to demonstrate their effectiveness (i.e., a high probability of correct selection (PCS)) and efficiency (i.e., a small total number of simulated observations). For instance, the OCBA (optimal computing budget allocation) algorithm in Chen et al. [6] and the EVI (expected value of information) procedures in Chick and Inoue [8, 9], under the Bayesian approach, attempt to allocate a given number of simulation budgets to maximize the posterior PCS, while fully sequential procedures such as Paulson's procedure in Paulson [34] and the KN family procedures in Kim and Nelson [21, 22] under the IZ formulation of frequentist approach allow for early elimination of inferior systems. All of these approaches show that they can achieve the objective of either high effectiveness or high efficiency, depending on the particular problem formulations. It is worth pointing out that common random number (CRN) techniques have been introduced in EVI and KN procedures to further improve the efficiency.

A correct implementation of CRN would introduce a positive correlation between any pair of systems, thereby reducing the variance in the difference between two systems. In addition to CRN, there are various variance reduction techniques (VRTs) developed in the simulation literature that can directly reduce the variance of each individual system, and therefore lead to an additional benefit when applied to solve R&S problems. As shown in Nelson and Staum [32], Tsai et al. [44] and Tsai and Nelson [43], the control variate (CV) technique has been successfully adapted to R&S procedures to obtain greater statistical efficiency as compared to ordinary sample-mean-based procedures. In particular, Nelson and Staum [32] proposed a screening and a two-stage selection procedure that can efficiently employ CV estimators. Tsai et al. [44] derived a combined approach that uses a CV screening procedure to remove inferior systems in the first stage and then employs a CV selection procedure to the surviving systems in the second stage. Tsai and Nelson [43] developed fully sequential selection procedures based on controlled sum (CS) estimators (which is a variation of CV estimators; see Section 3.1 for more details). In this paper, we consider more complicated integrated VRTs rather than simple CV methods, and incorporate these VRTs into a

fully sequential framework under the IZ formulation.

When combining various integrated VRTs in a fully sequential framework, we notice that some of the existing VRTs may not always be applicable as we investigate the raw sum of a system's outputs. Take the poststratified sampling (PS) method for example, where some strata may be empty (i.e., none of the simulated observations falls in a particular stratum) and the total sample size is small, which makes the PS estimator unsuitable at the beginning. However, as the sample size increases, the possibility of empty strata decreases. This motivates us to study the asymptotic properties of the newly designed, fully sequential procedures in a limiting regime in which the IZ parameter goes to zero. It is often useful to combine the CV and conditional expectation (CE) techniques (see Section 3.2 for more details). As illustrated in Minh [30], the CE estimator may not be applicable in some scenarios. Then, the application of partial conditional expectation (PCE) estimators seems more attractive and is thus more widely used. Roughly speaking, as we simulate a sequence of observations from one system, some of them can be applied with the CE technique and thus contribute to a CV+CE combined estimator, while some may not and thus contribute only to a CV estimator. Then, the PCE technique allows the use of both estimators in the elimination process, resulting an additional reduction in the total sample size in the procedure. The contributions of this paper are two-fold: (1) derive closed-form expressions of the variances of CV+CE combined and CV+PS combined estimators, respectively, and provide analytical results that can evaluate or quantify the efficiency improvement compared with the ordinary CV estimator; (2) based on that, design fully sequential procedures that can exploit the benefits of adopting the integrated VRTs into sequential R&S settings.

1.1 Literature Review

Our work is mainly related to two streams of simulation literature. The first concerns integrated VRT strategies in the simulation literature. Grant [17] considered the pairwise combinations of antithetic variates (AV), CV and PS in a general context, in which they found that AV+PS is infeasible theoretically, and CV+PS outperforms CV+AV based on the simulation experimental results for the stochastic shortest route problem. Their empirical results also indicated that the combined CV+PS technique works better than either CV or PS used individually. The combined scheme with CV and stratified sampling (SS) was studied by L'Ecuyer and Buist [28] within the content of a call center simulation model, where stratification was applied to uniform random numbers driving the simulation. Sabuncuoglu et al. [39] investigated the experimental performance of individual use of different VRTs, including CV, AV, PS and Latin hypercube sampling (LHS), as well as the combined CV+AV and CV+LHS schemes. They considered three classical simulation models, i.e., $M/M/1$ queueing, a serial production line, and (s, S) inventory control systems, and

the experimental results (for the single use of VRT) showed that CV exhibits the best performance followed by PS. They also found that the output techniques (i.e., CV and PS) are superior to the input techniques (i.e., AV and LHS), especially when the systems under examination become more complex. A framework for integrated use of different VRTs, including the input techniques of AV and LHS as well as the output techniques of CV and CE, was established in Avramidis and Wilson [2], providing sufficient conditions under which the combined schemes are superior to their constituent VRT. These individual VRTs and integrated variance reduction strategies were implemented and compared via simulation experiments on stochastic activity networks, and the experimental results also revealed that the CE technique delivers the best efficiency among the individual VRTs they examined. The CV+CE scheme was investigated by Ross and Lin [37] to estimate the expected delay in an $M/M/1$ queue. These CV+CE schemes in both of the aforementioned papers are adopted in our current work, which can be considered as a natural estimator corresponding to our CV+CE Combined Models I and II (see Sections 3.2.1 and 3.2.2). It is worthwhile noting that most of the existing works evaluated the performance of each combined method in specific stochastic systems and drew conclusions only based on the simulation experiments.

The second stream of simulation literature concerns fully sequential R&S procedures under IZ formulation. As mentioned before, the IZ formulation was first proposed by Bechhofer [3] (providing a single-stage procedure with a fixed sample size) and later incorporated into the fully sequential scheme by Paulson [34]. Kim and Nelson [21] generalized Paulson's result to unknown and unequal variance cases with a tight bound based on the results of Fabian [12]. Most fully sequential procedures, e.g., Paulson's procedure in Paulson [34] and the KN procedure in Kim and Nelson [21], take a single observation from each system still in contention at each stage (which is called the vector-at-a-time (VT) sampling rule in Jennison et al. [20] or the round-robin rule in Luo et al. [29]), and eliminate systems whenever they are statistically inferior. It has been shown that, in general, early elimination due to the fully sequential nature of simulations will greatly reduce the total computational effort compared with multi-stage procedures, such as the single-stage procedure in Bechhofer [3] and the two-stage procedure in Rinott [35]. More sophisticated sampling rules other than VT have demonstrated additional statistical efficiency, such as the asymptotic sampling rules in Jennison et al. [20] and the variance-dependent rules in Hong [19]. However, to evaluate the advantage of integrating various VRTs and to make a fair comparison with KN-like procedures, we focus on the VT sampling rules in this paper.

In addition to a sequence of papers that considered CV techniques in R&S procedures (e.g., Nelson and Staum [32], Tsai and Nelson [43] and Tsai et al. [44]), there is one more related work of Tsai and Kuo [42], which further exploited models combining CV and input techniques (i.e., CV+AV and CV+LHS). A summary of existing research work on VRTs is given in Table 1, which

Table 1: Summary of the research work on VRTs.

Papers	Contributions	VRTs used	Problems or procedures studied
Grant [17]	Comparison	CV+PS, CV+AV	Stochastic routing problem
L'Ecuyer and Buist [28]	Comparison	CV+SS	Call center system
Sabuncuoglu et al. [39]	Comparison	CV+AV, CV+LHS	$M/M/1$ queue, production line, and inventory system
Ross and Lin [37]	Comparison	CV+CE	$M/M/1$ queue
Avramidis and Wilson [2]	Comparison	Combinations among AV, LHS, CV and CE	Stochastic activity network
Nelson and Staum [32]	R&S	CV	Screening and two-stage selection
Tsai et al. [44]	R&S	CV	Combined screening and selection
Tsai and Nelson [43]	R&S	CV	Fully sequential procedure
Tsai and Kuo [42]	R&S	CV+AV, CV+LHS	Screening, two-stage, and fully sequential procedures
Current work	R&S	CV+CE, CV+PCE, and CV+PS	Fully sequential procedures

includes (i) comparative studies to test some of the VRTs under various experimental conditions; and (ii) specifically developed R&S procedures to exploit VRTs. Different from Tsai and Kuo [42], in this paper, we consider combined models of CV with output techniques (i.e., CV+CE and CV+PS), and design new fully sequential procedures when combined models may sometimes not be applicable (which motivates us to use CV+PCE). This research work is also motivated by the observation that output techniques can often obtain more statistical efficiency than input techniques, as shown in the related literature (e.g., Avramidis and Wilson [2], Grant [17], and Sabuncuoglu et al. [39]). We also note that the aforementioned CV+PS scheme in the literature assumes a fixed sample size and requires implementing the regression for each stratum. It is therefore not appropriate for direct use in fully sequential procedures (see Section 4.2 for more arguments). Such difficulties drive us to study statistical effectiveness and efficiency in an asymptotic regime in addition to the finite-time properties.

The rest of this article is organized as follows: In Section 2, we develop a general fully sequential procedure (FSP) without introducing the detailed notions of VRTs. A brief review of individual VRTs and the derivation of the combined models, especially CV+CE, are provided in Section 3. The details of the CV+PS combined model and its corresponding FSP are then given in Section 4. We also provide an extensive numerical study and a realistic illustration in Sections 5 and 6, respectively, and make some concluding remarks in Section 7. The proof, the details of the benchmark procedures, and the numerical results for FSP with CV+CE models are contained in the Appendix.

2 Overview of the Proposed Fully Sequential Procedure

Suppose that we intend to select the best system with the largest mean performance measure among a set of k system designs with unknown mean θ_i, $i = 1, 2, \ldots, k$. Under the IZ formulation, we further assume that $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, where $\delta > 0$ is the IZ parameter representing the practically significant difference in the expected performance that is worth detecting. The original unknown variance of system i is denoted by σ_i^2, which can be reduced to u_i^2 after successfully applying the CV technique, for instance, as shown in Tsai and Nelson [43]. In this paper, we consider the application of more sophisticated integrated VRTs to further reduce the variance to v_i^2, where $v_i^2 < u_i^2 < \sigma_i^2$, $i = 1, 2, \ldots, k$. Then, an FSP with finite-sample statistical validity that can exploit the aforementioned integrated VRTs, especially in the case of various CV+CE combined models, can be designed in a similar fashion as in Tsai and Nelson [43], which is presented in Appendix B.1.

Meanwhile, it is worthwhile noticing that the implementation of integrated VRTs is not always applicable as the implementation of CV. We have to admit that the uncertainty of whether being able to apply integrated VRTs or not could cause potential difficulties as we analyze the finite-time statistical validity, which drives us to study the asymptotic properties in the regime where the IZ parameter $\delta \to 0$. This asymptotic regime $\delta \to 0$ was also used by Kim and Nelson [22] in analyzing steady-state simulation observations and by Luo et al. [29] for designing fully sequential procedures in parallel computing environments. In the following subsection we introduce a new FSP that can employ CV+PCE estimators (the details for which are discussed in Section 3) and feature the variance updating mechanism. The statistical validity of the proposed procedure in the asymptotic regime (i.e., $\delta \to 0$) is shown in Appendix A.

2.1 Fully Sequential Procedure with CV+PCE

A major issue one faces when using the CE technique is that sometimes it is difficult to find a conditional variable for which the CE estimators can be obtained for all simulation replications. The following are examples where some observations from one system may not be applied with the CE technique: (a) we want to estimate the average annual salary of a country, and know exactly the average salary of all government officers (but not the rest of the population); (b) we want to estimate the expected waiting time for the G/G/1 queue with the first-in-first-out discipline as employing the Lindley equation. Minh [30] illustrated that the CE technique can only be applicable when the difference between the service time and interarrival time of the nth customer is greater than or equal to zero; (c) we want to estimate the expected waiting time for a queue where there are two classes of customers with different service time distributions (e.g., exponential and normal).

We use the number of customers waiting ahead in line when each customer just enters the system as the conditional variable. The CE technique can only be applied when the service time of the customer in process follows an exponential distribution (see Section 6.2 for more details); (d) we want to estimate the expected time to failure of a complex system that consists of several dependent components with various degradation mechanisms. The application of CE seems possible only when the component degradation is not observed or not evident in some system scenarios (Zille et al. [47]).

We propose using the CV+PCE technique which employs a CV+CE combined estimator when some replications from one system can be applied with the CE technique, while uses a pure CV estimator in some replications when the CE technique cannot be applied. For notational simplicity, let X_{ij} denote the original jth observation taken from system i, whose mean and variance are θ_i and σ_i^2, for $i = 1, 2, \ldots, k$. We then check whether the integrated VRT can be applied to that observation. If it can, then we obtain an outcome, denoted as $V_{i\ell}$, the ℓth observation after applying the integrated VRT (e.g., CV+CE). Otherwise, we obtain an outcome, denoted as U_{ij}, the jth observation after simply applying the CV technique. Both $V_{i\ell}$ and U_{ij} are assumed to have the same mean θ_i, but different variances, v_i^2 and u_i^2, respectively. Under some conditions (which are discussed in Section 3.2), we can make sure that $v_i^2 < u_i^2$ holds for all $i = 1, 2, \ldots, k$.

In designing fully sequential procedures, we assume that we take only one observation for each system still in contention at each stage. At stage r, let m_{ir} and n_{ir} denote the total number of observations obtained by CV and CV+CE, respectively. Then, m_{ir} and n_{ir} are non-decreasing random variables as r increases, and $m_{ir}+n_{ir} = r$. Note that the various VRTs will intend to reduce the variability of outputs without affecting the mean performance. To fully take the advantage of this, we continue updating the variance at each stage.

Let $U_i(\cdot)$, $U_i^2(\cdot)$, $V_i(\cdot)$ and $V_i^2(\cdot)$ be defined as follows,

$$U_i(m_{ir}) = \sum_{j=1}^{m_{ir}} U_{ij}, \quad \text{and} \quad U_i^2(m_{ir}) = \sum_{j=1}^{m_{ir}} U_{ij}^2, \tag{1}$$

$$V_i(n_{ir}) = \sum_{\ell=1}^{n_{ir}} V_{i\ell}, \quad \text{and} \quad V_i^2(n_{ir}) = \sum_{\ell=1}^{n_{ir}} V_{i\ell}^2. \tag{2}$$

Then the unbiased sample variance estimators when $m_{ir} \geq 2$ and $n_{ir} \geq 2$ are

$$S^2_{\mathrm{CV}}(i,r) = \frac{1}{m_{ir}-1}\left[U_i^2(m_{ir}) - \frac{1}{m_{ir}}\left(U_i(m_{ir})\right)^2\right], \tag{3}$$

$$S^2_{\mathrm{CV+CE}}(i,r) = \frac{1}{n_{ir}-1}\left[V_i^2(n_{ir}) - \frac{1}{n_{ir}}\left(V_i(n_{ir})\right)^2\right]. \tag{4}$$

When $m_{ir} = 0, 1$ or $n_{ir} = 0, 1$, we may artificially define $S^2_{\mathrm{CV}}(i,r) = 0$ or $S^2_{\mathrm{CV+CE}}(i,r) = 0$.

This should be fine in the asymptotic regime where we make the assumption that $m_{ir} \to \infty$ and $n_{ir} \to \infty$ as $r \to \infty$ (see Theorem 1 for a rigorous statement of the asymptotic regime). In **Step 0** of Procedure 1, we let the first-stage sample size $n_0 \geq 3$ to avoid a trivial case in which we cannot calculate the sample variance.

We are now ready to present the procedure. For simplicity of presentation, we here omit the detailed parameter settings, e.g., estimating the unknown multiplier parameter in a regression model, which is further discussed in Sections 3.2 and 5.2.1.

Procedure 1 (Fully Sequential Procedure with CV+PCE)**.**

Step 0. Setup: Select confidence level $1/k < 1 - \alpha < 1$, IZ parameter $\delta > 0$, and the first-stage sample size $n_0 \geq 3$. Let $a = -\log[2\alpha/(k-1)]$.

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. Let r be the observation counter. Set $r = n_0$. Take r samples from system i, resulting m_{ir} samples of $\{U_{ij}, j = 1, \ldots, m_{ir}\}$ and n_{ir} samples of $\{V_{i\ell}, \ell = 1, \ldots, n_{ir}\}$, where $m_{ir} + n_{ir} = r$. Let $U_i(m_{ir}) = U_i^2(m_{ir}) = 0$ and $V_i(n_{ir}) = V_i^2(n_{ir}) = 0$. If $m_{ir} > 0$ or $n_{ir} > 0$, then update $U_i(m_{ir})$, $U_i^2(m_{ir})$, $V_i(n_{ir})$ and $V_i^2(n_{ir})$ according to Equations (1) and (2).

Let $S^2_{\mathrm{CV}}(i, r) = 0$ and $S^2_{\mathrm{CV+CE}}(i, r) = 0$. If $m_{ir} \geq 2$ or $n_{ir} \geq 2$, then update $S^2_{\mathrm{CV}}(i, r)$ or $S^2_{\mathrm{CV+CE}}(i, r)$ according to Equations (3) and (4).

Step 2. Elimination: For any pair of systems i and h in I, $i \neq h$, compute

$$\hat{\sigma}^2_{ih}(r) = \frac{1}{r}\left[m_{ir} S^2_{\mathrm{CV}}(i, r) + n_{ir} S^2_{\mathrm{CV+CE}}(i, r) + m_{hr} S^2_{\mathrm{CV}}(h, r) + n_{hr} S^2_{\mathrm{CV+CE}}(h, r)\right], \quad (5)$$

$$Z_{ih}(r) = \frac{1}{r}\left[U_i(m_{ir}) + V_i(n_{ir}) - U_h(m_{hr}) - V_h(n_{hr})\right]. \quad (6)$$

Set $I^{\mathrm{old}} = I$. Let

$$I = I^{\mathrm{old}} \setminus \left\{ i \in I^{\mathrm{old}} : Z_{ih}(r) < \min\left\{0, -\frac{a\hat{\sigma}^2_{ih}(r)}{\delta r} + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\mathrm{old}} \text{ and } h \neq i \right\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 3. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r + 1$ and take the rth sample from system $i \in I$. If the rth sample is from $U_{i\cdot}$, then update $m_{ir} = m_{i,r-1} + 1$, $U_i(m_{ir}) = U_i(m_{ir} - 1) + U_{i,m_{ir}}$ and $U_i^2(m_{ir}) = U_i^2(m_{ir} - 1) + U^2_{i,m_{ir}}$. If the rth sample is from $V_{i\cdot}$, then update $n_{ir} = n_{i,r-1} + 1$, $V_i(n_{ir}) = V_i(n_{ir} - 1) + V_{i,n_{ir}}$ and $V_i^2(n_{ir}) = V_i^2(n_{ir} - 1) + V^2_{i,n_{ir}}$. Also update $S^2_{\mathrm{CV}}(i, r)$ or $S^2_{\mathrm{CV+CE}}(i, r)$ according to Equations (3) or (4). Then go to Step 2.

Even though Procedure 1 is designed for integrating the CV and PCE techniques, it can be easily generalized to employ other combined VRT models, with specifying other parameters in **Step 0** and **Step 1** as well as the particular variance and mean-difference estimators in Equations (5) and (6) in **Step 2**. For instance, we consider three different CV+CE combined models in Section 3.2, which can also be applied in the same fashion as in Procedure 1. We now state the main theorem as follows.

Theorem 1. *Let U_{ij} and $V_{i\ell}$ denote the jth and ℓth observations from system i, $i = 1, 2, \ldots, k$, that being applied CV and CV+CE techniques, respectively. We assume that systems i and h are independently simulated for any $i \neq h$. Let $\theta_i = \mathrm{E}[U_{ij}] = \mathrm{E}[V_{i\ell}]$ be the unknown mean of system i and $u_i^2 = \mathrm{Var}[U_{ij}]$ and $v_i^2 = \mathrm{Var}[V_{i\ell}]$ be the unknown but finite variances with $v_i^2 < u_i^2$. Moreover, we assume that U_{ij} and $V_{i\ell}$ are independent of each other for any j and ℓ. Without loss of generality, we let $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, where δ is the IZ parameter. Let the first-stage sample size $n_0 = n_0(\delta)$ be a function of δ such that $n_0 \to \infty$ and $\delta^2 n_0 \to 0$ as $\delta \to 0$. Let m_{ir} and n_{ir} denote the sample size of $\{U_{ij}, j = 1, \ldots, m_{ir}\}$ and $\{V_{i\ell}, \ell = 1, \ldots, n_{ir}\}$ at stage r, which is also a function of δ. We assume that $m_{ir} \to \infty$ and $n_{ir} \to \infty$, and $m_{ir}/r \to p_i$ and $n_{ir}/r \to 1 - p_i$, where p_i is some unknown constant in $[0, 1]$, as $\delta \to 0$ (implying that $r \to \infty$). Then, as $\delta \to 0$, Procedure 1 selects system 1 as the best with a probability at least $1 - \alpha$.*

The proof follows a similar argument as in Luo et al. [29]. The main step is to construct a modified stochastic process of $\widetilde{Z}_{ih}(\cdot)$ and show that it converges to a Brownian motion process, for which the detailed derivation is presented in Appendix A.

3 VRTs and Combined Models of CV and CE

The three classes of VRTs considered in this paper are control variate (CV), conditional expectation (CE) and poststratified sampling (PS) techniques. They all share the same feature where one or multiple auxiliary random variables must be identified before implementation. CV is generally applicable and can achieve efficiency gains by exploiting the intrinsic linear relationship between the output and the selected input random variables (called control variables) whose expectations are known. The choice of appropriate control variables is essential to realizing the benefits of variance reduction. The idea of CE is to replace the basic estimator by an expectation conditioning on a concomitant random variable (called a conditional variable) that remains available during the course of the simulation. Although the application of CE is problem-dependent, it can guarantee the effectiveness of variance reduction if implemented. When employing the PS technique, instead of sampling conditionally from the specified stratum, we simply perform ordinary independent sampling and then classify the observations into the appropriate stratum defined in terms of an

auxiliary random variable (called a stratification variable) whose exact distribution is known. The stratified sampling estimator is employed to eliminate sampling variability across strata (without affecting the intra-stratum variability), thus delivering a smaller variance compared to the ordinary sample mean. In this section, we mainly focus on the CV, CE techniques and their combined models. The results of combining CV and PS are deferred to Section 4.

Let Y_{ij} represent the simulation output from the jth replication of system i, for $i = 1, 2, \ldots, k$. The ordinary sample-mean-based R&S procedures assume that

$$Y_{ij} = \theta_i + \eta_{ij},$$

where η_{ij} is the error term, which is assumed to be an independently and identically distributed (i.i.d.) $\mathrm{N}(0, \sigma_i^2)$ random variables with σ_i^2 unknown for all i and j. The standard estimator of θ_i across n simulation replications is the sample mean

$$\bar{Y}_i(n) = \frac{1}{n} \sum_{j=1}^{n} Y_{ij},$$

which is unbiased and has the variance σ_i^2/n.

3.1 Control Variates and Conditional Expectations

We first review the linear CV model (referred to as Model 0), and the following description is primarily based on Tsai and Nelson [43]. Let Y_{ij} and $\mathbf{C}_{ij}$ represent the simulation outputs and functions of inputs, respectively, from the jth replication of system i for $i = 1, 2, \ldots, k$. Suppose that the intrinsic relationship between them can be described by the following linear model:

$$Y_{ij} = \theta_i + (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^T \boldsymbol{\beta}_i + \epsilon_{ij},$$

where θ_i is the unknown true mean of Y_{ij}. The $q_i \times 1$ vector $\mathbf{C}_{ij}$ is called the control variable and is assumed to be multivariate normal with a known mean vector $\boldsymbol{\mu}_i$. The multiplier $\boldsymbol{\beta}_i$ is a $q_i \times 1$ vector of unknown constants, and ϵ_{ij} is the error term that is assumed to be an i.i.d. $\mathrm{N}(0, \tau_i^2)$ random variable. The Y_{ij} are i.i.d. $\mathrm{N}(\theta_i, \sigma_i^2)$ random variables with both θ_i and σ_i^2 unknown and perhaps unequal.

We apply a linear regression analysis on Model 0 to produce the following linear CV point estimator of θ_i:

$$\widehat{\theta}_{\mathrm{CV}}(i, n) \;=\; \bar{Y}_i(n) - \left(\bar{\mathbf{C}}_i(n) - \boldsymbol{\mu}_i\right)^T \widehat{\boldsymbol{\beta}}_i(n),$$

where

$$\bar{Y}_i(n) = \frac{1}{n}\sum_{j=1}^{n} Y_{ij} \quad \text{and} \quad \bar{\mathbf{C}}_i(n) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{C}_{ij},$$

which represent the sample means of the outputs and control variables, and

$$\widehat{\boldsymbol{\beta}}_i(n) = \mathbf{S}_{\mathbf{C}_i}^{-1}(n)\mathbf{S}_{\mathbf{C}_i Y_i}(n) \tag{7}$$

where $\mathbf{S}_{\mathbf{C}_i}(n)$ is the sample variance-covariance matrix of $\mathbf{C}_{ij}$, and $\mathbf{S}_{\mathbf{C}_i Y_i}(n)$ is the sample covariance vector between $\mathbf{C}_{ij}$ and Y_{ij}. The expectation and variance of $\widehat{\theta}_{\mathrm{CV}}(i,n)$ can be shown as follows:

$$\mathrm{E}[\widehat{\theta}_{\mathrm{CV}}(i,n)] = \theta_i \quad \text{and} \quad \mathrm{Var}[\widehat{\theta}_{\mathrm{CV}}(i,n)] = \left(\frac{n-2}{n-q_i-2}\right)\frac{\tau_i^2}{n}, \tag{8}$$

where $\tau_i^2 = (1-R_i^2)\sigma_i^2$ and R_i^2 is the square of the multiple correlation coefficient between Y_{ij} and $\mathbf{C}_{ij}$. The term $(n-2)/(n-q_i-2)$ is known as the loss ratio, which quantifies the efficiency loss due to the estimation of the unknown multiplier $\boldsymbol{\beta}_i$.

The standard CV estimator is statistically more efficient than the ordinary sample mean, but it also requires more computational effort (i.e., the implementation of a linear regression). Tsai and Nelson [43] proposed a variation of CV estimators, which we call controlled sum (CS) estimators, that can be appropriately incorporated into an FSP. We need to collect preliminary-stage samples $(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0$ to compute $\widehat{\boldsymbol{\beta}}_i(m_0)$ for each system $i = 1, 2, \ldots, k$ (based on Equation (7)). For any non-negative integers a, b, with $b > a+1$, and $q_i \times 1$ vector $\widehat{\boldsymbol{\beta}}_i(m_0)$ as just defined, the *controlled sample mean* from the $(a+1)$st sample to the bth sample is defined as

$$\bar{Y}_i[a,b] = \frac{1}{b-a}\sum_{j=a+1}^{b}\left[Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^T\,\widehat{\boldsymbol{\beta}}_i(m_0)\right].$$

For all $i \neq h$, define the *controlled sample variance*, $S_{ih}^2[a,b]$, as

$$\frac{1}{b-a-1}\sum_{j=a+1}^{b}\left[Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^T\,\widehat{\boldsymbol{\beta}}_i(m_0) - Y_{hj} + (\mathbf{C}_{hj} - \boldsymbol{\mu}_h)^T\,\widehat{\boldsymbol{\beta}}_h(m_0) - \bar{Y}_i[a,b] + \bar{Y}_h[a,b]\right]^2. \tag{9}$$

Suppose we also specify a first-stage sample size n_0. We then take additional observations $(Y_{ij}, \mathbf{C}_{ij}), j = m_0+1, m_0+2, \ldots, m_0+n_0$ from which the controlled sample mean $\bar{Y}_i[m_0, m_0+n_0]$ and the controlled sample variance $S_{ih}^2[m_0, m_0+n_0]$ can be derived. If the linear CV model holds then it can

be shown that

$$\mathrm{E}\left[\bar{Y}_i[m_0, m_0+n_0]\right] = \theta_i \quad \text{and} \quad \mathrm{Var}\left[\bar{Y}_i[m_0, m_0+n_0]\right] = \left(\frac{m_0-2}{m_0-q_i-2}\right)\frac{\tau_i^2}{n_0}. \tag{10}$$

See Tsai and Nelson [43] for some guidelines by which to choose an appropriate value of m_0.

Besides the CV technique, in some simulation models, we may be able to exploit their special problem structure or properties, and then identify an auxiliary random vector (or a random scalar) $\mathbf{X}_{ij}$, which is called the conditional variable, such that the conditional expectation $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$ can possibly be evaluated analytically or numerically for every possible value of $\mathbf{X}_{ij}$. Then, without considering CV technique, the standard CE estimator of θ_i across n simulation replications is the sample mean

$$\widehat{\theta}_{\mathrm{CE}}(i,n) = \frac{1}{n}\sum_{j=1}^{n}\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$$

for $i = 1, 2, \ldots, k$, which can be shown to be unbiased and has the variance $\mathrm{Var}[\widehat{\theta}_{\mathrm{CE}}(i,n)] = n^{-1}\left(\sigma_i^2 - \mathrm{E}[\mathrm{Var}(Y_{ij}|\mathbf{X}_{ij})]\right)$ (see p.106 and p.118 of Ross [36]). We can immediately see that variance reduction can be achieved (i.e., $\mathrm{Var}[\widehat{\theta}_{\mathrm{CE}}(i,n)] \leq \mathrm{Var}[\bar{Y}_i(n)]$) because $\mathrm{E}[\mathrm{Var}(Y_{ij}|\mathbf{X}_{ij})] \geq 0$.

3.2 Combined Models of CV and CE

When both the control variates and conditional expectation techniques are applicable in the simulation experiment, we expect that additional variance reduction can be achieved. Specifically, we consider the following three combined schemes: (I) replacing the simulation output and control variables with their conditional expectations, $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$ and $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$, (II) using only the conditional expectation of the simulation output, $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$, and (III) using only the conditional expectation of the control variables, $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$.

Although these combined models might only be applicable to specific problems or scenarios, in the following subsections, we derive the analytical results to identify the conditions by which each model can outperform the others (as well as a pure CV or CE model). To simplify the analytical comparison between different combined models, we may require the assumption of multivariate normality among the output and input variables (or their functions). This might be reasonable because in a simulation experiment we collect observations across multiple replications and then take average of them, which implies that a multivariate version of the central limit theorem can be applied (Nelson [31], p.231). It should also be noted that in subsections 3.2.1–3.2.3, we assume that the conditional expectations $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$ and $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$ can be obtained analytically for *every* possible value of $\mathbf{X}_{ij}$. Then, a statistically valid FSP that can efficiently employ these CV+CE combined models can be designed in a similar fashion as in Tsai and Kuo [42] and Tsai and Nelson

[43], which is presented in Appendix B.1. As discussed in Section 2.1, this assumption can be relaxed to make the combined models more widely applicable since it may not be possible to apply the CE technique in some simulation replications.

3.2.1 CV+CE Combined Model I

This subsection presents a model for combining CV and CE in such a way that both the conditional expectation of outputs (i.e., $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$) and control variables (i.e., $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$) can be evaluated either analytically or numerically. The Combined Model I can be described as follows:

For each system $i = 1, 2, \ldots k$,

$$\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}] \quad = \quad \theta_i + (\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}] - \boldsymbol{\mu}_i)^T \boldsymbol{\beta}_i^{(1)} + \epsilon_{ij}^{(1)}.$$

The distributions and relationship we assume for the conditional expectation of outputs and controls are similar to those of Model 0. We then apply a linear regression to provide the point estimator of θ_i that can exploit CV and CE:

$$\widehat{\theta}_{\mathrm{CV+CE}}^{(1)}(i,n) = \bar{Y}_i^{(1)}(n) - \left(\bar{\mathbf{C}}_i^{(1)}(n) - \boldsymbol{\mu}_i\right)^T \widehat{\boldsymbol{\beta}}_i^{(1)}(n) \tag{11}$$

where

$$\bar{Y}_i^{(1)}(n) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{E}[Y_{ij}|\mathbf{X}_{ij}],\ \bar{\mathbf{C}}_i^{(1)}(n) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}] \ \text{ and } \ \widehat{\boldsymbol{\beta}}_i^{(1)}(n) = \mathbf{S}^{-1}_{\bar{\mathbf{C}}_i^{(1)}}(n)\mathbf{S}_{\bar{\mathbf{C}}_i^{(1)}\bar{Y}_i^{(1)}}(n) \tag{12}$$

where $\mathbf{S}^{-1}_{\bar{\mathbf{C}}_i^{(1)}}(n)$ is the sample variance-covariance matrix of $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$ and $\mathbf{S}_{\bar{\mathbf{C}}_i^{(1)}\bar{Y}_i^{(1)}}(n)$ is the sample covariance vector between $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$ and $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$. We can show that $\widehat{\theta}_{\mathrm{CV+CE}}^{(1)}(i,n)$ is an unbiased estimator of θ_i and its variance is

$$\mathrm{Var}\left[\widehat{\theta}_{\mathrm{CV+CE}}^{(1)}(i,n)\right] \quad = \quad \left(\frac{n-2}{n-q_i-2}\right)\left(1-R_i'^2\right)\frac{\mathrm{Var}\left[\mathrm{E}\left[Y_{ij}|\mathbf{X}_{ij}\right]\right]}{n} \tag{13}$$

$$\quad = \quad \left(\frac{n-2}{n-q_i-2}\right)\left(1-R_i'^2\right)\frac{\mathrm{Var}[Y_{ij}] \times R^2_{Y,\mathbf{X}}}{n} \tag{14}$$

where $R_i'^2$ is the square of the multiple correlation coefficient between $\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}]$ and $\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]$, and $R^2_{Y,\mathbf{X}}$ is the square of the multiple correlation coefficient between Y_{ij} and $\mathbf{X}_{ij}$. Equation (13) follows because we assume that Combined Model I holds. Equation (14) follows only when we also assume that the joint distribution of $\{Y_{ij}, \mathbf{X}_{ij}^T\}$ is multivariate normal, in which case we can obtain $\mathrm{Var}\left[\mathrm{E}\left[Y_{ij}|\mathbf{X}_{ij}\right]\right] = \mathrm{Var}[Y_{ij}] \times R^2_{Y,\mathbf{X}}$. In addition, if we also assume that Model 0 holds and compare Equation (14) with the variance term of Equation (8), then we can obtain $\mathrm{Var}\left[\widehat{\theta}_{\mathrm{CV+CE}}^{(1)}(i,n)\right] \leq$

Var $\left[\widehat{\theta}_{\mathrm{CV}}(i,n)\right]$ as long as the following condition is satisfied:

$$R^2_{Y,\mathbf{X}}\left(1-R'^2_i\right) \leq 1-R^2_{Y,\mathbf{C}}.$$

This implies that we prefer using a set of conditional variables that has a small correlation with the simulation output Y. As expected, we also want to obtain a significant correlation between the two conditional expectation terms of Combined Model I. Avramidis and Wilson [2] examined the efficiency of Combined Model I in the experiments on stochastic activity networks and showed that $\widehat{\theta}^{(1)}_{\mathrm{CV+CE}}(i,n)$ asymptotically dominates $\widehat{\theta}_{\mathrm{CV}}(i,n)$ in terms of estimator variance under some specific assumptions. They considered the effect of variance reduction and did not apply the combined model in a fully sequential selection procedure. By contrast, in the following subsection, we demonstrate that in the scenario of finite samples, Combined Model I is superior to Combined Model II in terms of statistical efficiency (through an analytical comparison), and then show that Combined Model II is very likely to work better than Model 0 (through a simple numerical example). In this way, we can indirectly show that Combined Model I should be superior to Model 0. Furthermore, following similar derivations as in Theorem 3 of Avramidis and Wilson [2] (i.e., directly comparing Equation (13) with Var $\left[\widehat{\theta}_{\mathrm{CE}}(i,n)\right]$), we can obtain Var $\left[\widehat{\theta}^{(1)}_{\mathrm{CV+CE}}(i,n)\right] \leq$ Var $\left[\widehat{\theta}_{\mathrm{CE}}(i,n)\right]$ as long as the sample size n is not too small (i.e., $n \geq q_i/R'^2_i + 2$). It should also be noted that assuming that the joint distribution of $\{Y_{ij}, \mathbf{C}^T_{ij}, \mathbf{X}^T_{ij}\}$ is nonsingular multivariate normal implies that $\{Y_{ij}, \mathbf{C}^T_{ij}\}$ (for Model 0 to hold), $\{Y_{ij}, \mathbf{X}^T_{ij}\}$ (for Equation (14) to hold), and $\{\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}], \mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]^T\}$ (for Combined Model I to hold) are all multivariate normally distributed (see Theorems 3.3.1 and 3.3.4 of Tong [41]).

3.2.2 CV+CE Combined Model II

Since the application of CE is problem-dependent, in some cases we may not be able to find an appropriate conditional variable $\mathbf{X}_{ij}$ that allows taking expectation for both the output and control variables. Therefore, in this subsection we introduce another combined model in which only the conditional expectation of the outputs can be evaluated analytically. The Combined Model II can be described as follows:

For each system $i = 1, 2, \ldots k$,

$$\mathrm{E}\left[Y_{ij}|\mathbf{X}_{ij}\right] \;=\; \theta_i + \left(\mathbf{C}_{ij} - \boldsymbol{\mu}_i\right)^T \boldsymbol{\beta}^{(2)}_i + \epsilon^{(2)}_{ij}.$$

The distributions and relationships we assume for $\mathrm{E}\left[Y_{ij}|\mathbf{X}_{ij}\right]$, $\mathbf{C}_{ij}$, and $\epsilon^{(2)}_{ij}$ are similar to those of Combined Model I. Based on Combined Model II, the point estimator $\widehat{\theta}^{(2)}_{\mathrm{CV+CE}}(i,n)$ can be defined in a similar fashion as in Equation (11). We can also justify that $\widehat{\theta}^{(2)}_{\mathrm{CV+CE}}(i,n)$ is unbiased and its

variance is

$$\text{Var}\left[\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n)\right] = \left(\frac{n-2}{n-q_i-2}\right)\left(1-R^2_{\text{E}[Y|\mathbf{X}],\mathbf{C}}\right)\frac{\text{Var}\left[\text{E}\left[Y_{ij}|\mathbf{X}_{ij}\right]\right]}{n} \quad (15)$$

$$= \left(\frac{n-2}{n-q_i-2}\right)\left(1-R^2_{\text{E}[Y|\mathbf{X}],\mathbf{C}}\right)\frac{\text{Var}[Y_{ij}]\times R^2_{Y,\mathbf{X}}}{n}. \quad (16)$$

Similar to the previous model, Equation (15) follows because we assume that Combined Model II holds. Equation (16) holds only when we also assume that the joint distribution of $\{Y_{ij}, \mathbf{X}^T_{ij}\}$ is multivariate normal. We now proceed to compare the statistical efficiency of $\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n)$ with that of $\widehat{\theta}_{\text{CV}}(i,n)$ and $\widehat{\theta}^{(1)}_{\text{CV+CE}}(i,n)$. For the convenience of illustration, let us assume that the conditional variable X_{ij} is a scalar (instead of a vector). Since $\text{E}[Y_{ij}|X_{ij}]$ is a linear function of X_{ij} (when assuming that $\{Y_{ij}, X_{ij}\}$ is bivariate normally distributed), Equation (16) can be simplified as follows

$$\left(\frac{n-2}{n-q_i-2}\right)\left(1-R^2_{X,\mathbf{C}}\right)\frac{\text{Var}[Y_{ij}]\times R^2_{Y,X}}{n}.$$

Similar to the previous derivation, if we also assume that Model 0 holds, then we can obtain $\text{Var}\left[\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n)\right] \leq \text{Var}\left[\widehat{\theta}_{\text{CV}}(i,n)\right]$ as long as the following condition is satisfied:

$$R^2_{Y,X}\left(1-R^2_{X,\mathbf{C}}\right) \leq 1-R^2_{Y,\mathbf{C}}. \quad (17)$$

The $R^2_{Y,X}$ term represents the effect of applying CE to the output, and the $R^2_{X,\mathbf{C}}$ term represents the effect of using CV to explain the variability of outputs. This derivation implies that we should choose a conditional variable that has a small correlation with the output, but is strongly correlated with the vector of the control variables. We assume $q_i = 1$ to simplify the following analysis. Langford et al. [26] derived a theoretically valid inequality representing the lower and upper bound of $R^2_{Y,C}$ for any general distribution (when given a specified value of $R^2_{Y,X}$ and $R^2_{X,C}$).[1] However, we then find that these bounds might be too conservative to provide conclusive results. Instead, we simply implement simple numerical experiments to check whether the condition (17) can easily hold or not. In each trial, we generate 300 observations of multivariate normal random vector $\{Y, X, C\}$ that conform to the pre-specified settings of $R^2_{Y,X}$ and $R^2_{X,C}$. We first sample $X \sim N(0,1)$ and then, based on this, generate bivariate normal random vector $\{Y, X\}$ and $\{X, C\}$ with zero mean and unit variance by the linear transformation of i.i.d. normal random variables (see Section 2.3 of Chapter 11 of Devroye [11]). Table 2 then lists the minimum, average, and

[1]The inequality is given as follows:

$$R_{Y,X}R_{X,C} - \sqrt{(1-R^2_{Y,X})(1-R^2_{X,C})} \leq R_{Y,C} \leq R_{Y,X}R_{X,C} + \sqrt{(1-R^2_{Y,X})(1-R^2_{X,C})}.$$

Table 2: Minimum, average, and maximum values of the estimated $R^2_{Y,C}$ among 100 trials.

	$R^2_{X,C}=0.1$			$R^2_{X,C}=0.3$			$R^2_{X,C}=0.5$			$R^2_{X,C}=0.7$			$R^2_{X,C}=0.9$		
$R^2_{Y,X}$	min	avg	max	min	avg	max	min	avg	max	min	avg	max	min	avg	max
0.1	0	0.01	0.06	0	0.03	0.09	0.01	0.05	0.12	0.02	0.07	0.15	0.03	0.09	0.19
0.3	0	0.03	0.11	0.04	0.09	0.18	0.08	0.15	0.24	0.13	0.21	0.31	0.18	0.27	0.38
0.5	0.01	0.05	0.14	0.08	0.15	0.26	0.16	0.25	0.35	0.25	0.35	0.45	0.35	0.45	0.55
0.7	0.02	0.07	0.16	0.12	0.21	0.32	0.25	0.35	0.46	0.39	0.49	0.58	0.54	0.63	0.71
0.9	0.03	0.09	0.19	0.18	0.27	0.38	0.35	0.45	0.54	0.55	0.63	0.70	0.76	0.81	0.85

maximum of the estimated $R^2_{Y,C}$ among 100 trials. We can see that the condition (17) can be satisfied in almost all scenarios, except for the extreme cases when we look at the maximum $R^2_{Y,C}$ (among the 100 trials) with the setting of $R^2_{Y,X}=0.9$ and $R^2_{X,C}=0.1$ or 0.3. Even under these worst cases, the efficiency loss for employing Combined Model II is not significant (i.e., the ratio between $R^2_{Y,X}(1-R^2_{X,C})$ and $(1-R^2_{Y,C})$ is very close to one). We next compare Combined Model II with Combined Model I. To simplify the illustration, we also assume that the conditional variable is a scalar, $q_i=1$ and $\{Y_{ij},X_{ij}\}$ is bivariate normally distributed. We can then have $R_i'^2 = R^2_{X,\mathrm{E}[C|X]}$ and $R^2_{\mathrm{E}[Y|X],C} = R^2_{X,C}$ because $\mathrm{E}[Y_{ij}|X_{ij}]$ is a linear function of X_{ij}. Recall that $R^2_{X,\mathrm{E}[C|X]} = \frac{\mathrm{Cov}^2[X,\mathrm{E}[C|X]]}{\mathrm{Var}[X]\cdot\mathrm{Var}[\mathrm{E}[C|X]]}$ and $R^2_{X,C} = \frac{\mathrm{Cov}^2[X,C]}{\mathrm{Var}[X]\cdot\mathrm{Var}[C]}$. We can immediately see that $R^2_{X,\mathrm{E}[C|X]} \geq R^2_{X,C}$ because $\mathrm{Cov}\,[X,C]=\mathrm{Cov}\,[X,\mathrm{E}[C|X]]$ (see Chapter 4 of Casella and Berger [5]) and $\mathrm{Var}\,[\mathrm{E}[C|X]] \leq \mathrm{Var}[C]$ (a direct result when using conditional expectation). As a consequence, we can have $\mathrm{Var}\left[\widehat{\theta}^{(1)}_{\mathrm{CV+CE}}(i,n)\right] \leq \mathrm{Var}\left[\widehat{\theta}^{(2)}_{\mathrm{CV+CE}}(i,n)\right]$ when comparing Equation (14) with Equation (16).

3.2.3 CV+CE Combined Model III

The other combined model is used to obtain the conditional expectation of the control variables, which can be described as follows (denoted as Combined Model III):

For each system $i=1,2,\ldots k$,

$$Y_{ij} \;=\; \theta_i + (\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}]-\boldsymbol{\mu}_i)^T\boldsymbol{\beta}_i^{(3)} + \epsilon_{ij}^{(3)}.$$

Similar to the previous derivation, a regression analysis can be applied to this model to yield the point estimator $\widehat{\theta}^{(3)}_{\mathrm{CV+CE}}(i,n)$. Assuming that the Combined Model III holds, we can justify that

$\widehat{\theta}^{(3)}_{\text{CV+CE}}(i,n)$ is unbiased and its variance is

$$\begin{aligned} \text{Var}\left[\widehat{\theta}^{(3)}_{\text{CV+CE}}(i,n)\right] &= \left(\frac{n-2}{n-q_i-2}\right)\left(1-R^2_{Y,\,\text{E}[\mathbf{C}|\mathbf{X}]}\right)\frac{\text{Var}\left[Y_{ij}\right]}{n} \quad (18)\\ &= \left(\frac{n-2}{n-q_i-2}\right)\left(1-R^2_{Y,\mathbf{X}}\right)\frac{\text{Var}[Y_{ij}]}{n}. \quad (19) \end{aligned}$$

Equation (18) follows because we assume that Combined Model III holds. Equation (19) holds because $\text{E}\left[\mathbf{C}|\mathbf{X}\right]$ becomes a linear function of $\mathbf{X}$ when we also assume that the joint distribution of $\{\mathbf{C}^T_{ij}, \mathbf{X}^T_{ij}\}$ is multivariate normal (see Theorem 3.3.4 of Tong [41]). As a consequence, we can obtain $\text{Var}\left[\widehat{\theta}^{(3)}_{\text{CV+CE}}(i,n)\right] \leq \text{Var}\left[\widehat{\theta}_{\text{CV}}(i,n)\right]$ as long as the condition is satisfied: $R^2_{Y,\mathbf{X}} \geq R^2_{Y,\mathbf{C}}$. However, this condition might not be satisfied in general because we always prefer to choose control variables that have a strong linear association with the outputs when considering Model 0. It should be noted that any *input* random variables or variables that are generated by the simulation with known expectation can be chosen as control variables.

4 Control Variates and Poststratified Sampling (CV+PS)

We start by introducing the concept of stratified sampling (SS), in which we discover the difficulty related to the implementation into the fully sequential R&S framework, and therefore it motivates us to consider the poststratified sampling (PS) technique. The general idea of SS is to divide the sample space into L disjoint strata, and then within each stratum simple random sampling is applied (with a sample size determined a priori). Theoretically, the efficiency improvement of SS tends to increase as the number of strata increases. We need to identify a stratification variable whose distribution is known exactly and let D_{ij} denote its value for the jth replication of system i. The stratified estimator of θ_i can be obtained by computing the summation (over all L strata) of the product between the probability that the variable D_{ij} belongs to an individual stratum and its corresponding sample mean of Y_{ij}.

Before the implementation of SS technique, we need to decide how to construct the strata (i.e., partition the range of D into L intervals) and determine what fraction of the samples should be allocated to each stratum. To deal with the first issue, we have two different schemes: the equal-probability intervals and Sethi's optimal points of stratification (Sethi [40]). The other issue regarding the allocation fractions can also be handled using two different schemes: proportional allocation and optimal allocation (also called Neyman allocation, see, e.g., Cochran [10] for instance). However, the optimal allocation scheme is not directly applicable because its computation requires prior information on the stratum's standard deviations, which are, in general, unknown.

4.1 Combined Model of CV and PS

We first introduce the CV+SS combined model, which provides the fundamental basis required for the development of the target CV+PS combined estimator. Let A_ℓ, $\ell = 1, 2, \ldots, L$, denote stratum ℓ for a stratification variable D, and let $Y_{ij}^{(\ell)}$ have the distribution of Y_{ij} conditional on $D_{ij} \in A_\ell$. Moreover, let $\theta_i^{(\ell)} = \mathrm{E}[Y_{ij}^{(\ell)}] = \mathrm{E}[Y_{ij}|D_{ij} \in A_\ell]$. We also let $\mathbf{C}_{ij}^{(\ell)}$ have the distribution of $\mathbf{C}_{ij}$ conditional on $D_{ij} \in A_\ell$ with a known expected value $\boldsymbol{\mu}_i^{(\ell)} = \mathrm{E}[\mathbf{C}_{ij}^{(\ell)}] = \mathrm{E}[\mathbf{C}_{ij}|D_{ij} \in A_\ell]$. We let $N_i^{(\ell)}$ denote the number of observations $\{Y_{ij}, \mathbf{C}_{ij}, D_{ij}\}$ drawn from stratum A_ℓ (i.e., $N_i^{(\ell)} = \sum_{j=1}^{n} \mathbf{1}\{D_{ij} \in A_\ell\}$), where n represents the total sample size (i.e., $n = \sum_{\ell=1}^{L} N_i^{(\ell)}$). The value of $N_i^{(\ell)}$ is determined in advance, and simple random sampling is employed in each stratum A_ℓ. The CV+SS combined model can then be described as follows:

For each system $i = 1, 2, \ldots k$, and each stratum A_ℓ, $\ell = 1, 2, \ldots, L$,

$$Y_{ij}^{(\ell)} \;\; = \;\; \theta_i^{(\ell)} + \left(\mathbf{C}_{ij}^{(\ell)} - \boldsymbol{\mu}_i^{(\ell)}\right)^T \boldsymbol{\beta}_i^{(\ell)} + \epsilon_{ij}^{(\ell)}$$

where $\mathbf{C}_{ij}^{(\ell)}$ is the $q_i \times 1$ vector of control variables within stratum A_ℓ, following a multivariate normal distribution, while $\{\epsilon_{ij}^{(\ell)},\ j = \ 1, 2, \ldots, N_i^{(\ell)}\}$ is a set of i.i.d. $\mathrm{N}\left(0, (\tau_i^{(\ell)})^2\right)$ random variables, with $(\tau_i^{(\ell)})^2 = (1 - (R_i^{(\ell)})^2) \times \mathrm{Var}[Y_{ij}|D_{ij} \in A_\ell]$. Note that $(R_i^{(\ell)})^2$ is the square of the multiple correlation coefficient between $Y_{ij}^{(\ell)}$ and $\mathbf{C}_{ij}^{(\ell)}$, which is also called the partial correlation coefficient; see Theorem 3.4.3 of Tong [41]. In each stratum, the control variables $\{\mathbf{C}_{ij}^{(\ell)},\ j = \ 1, 2, \ldots, N_i^{(\ell)}\}$ are also i.i.d., and they are independent of $\{\epsilon_{ij}^{(\ell)},\ j = \ 1, 2, \ldots, N_i^{(\ell)}\}$.

A natural and intuitive way to exploit both CV and SS is to compute the standard CV estimator within each stratum and then produce a weighted mean (e.g., Grant [17] and L'Ecuyer and Buist [28]). However, this computational scheme is not appropriate for fully sequential procedures because it requires implementing a linear regression whenever a new observation is obtained (i.e., in every sampling stage). Therefore, we incorporate the concept of controlled-sum estimators into the stratified sampling context. We first collect preliminary-stage samples $(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0$ to compute $\widehat{\boldsymbol{\beta}}_i(m_0)$ for each system $i = 1, 2, \ldots, k$. Based on the above CV+SS combined model, the unbiased estimator of $\theta_i^{(\ell)}$ can be formulated as follows:

$$\widehat{Y}^{(\ell)}(i, N_i^{(\ell)}) = \frac{1}{N_i^{(\ell)}} \sum_{j=1}^{N_i^{(\ell)}} \left[Y_{ij}^{(\ell)} - (\mathbf{C}_{ij}^{(\ell)} - \boldsymbol{\mu}_i^{(\ell)})^T \widehat{\boldsymbol{\beta}}_i(m_0)\right].$$

Let $W_i^{(\ell)} = \Pr\{D_{ij} \in A_\ell\}$ denote the probability that the stratification variable D_{ij} belongs to the

ℓth stratum A_ℓ. A natural unbiased estimator of θ_i can then be given as follows:

$$\widehat{\theta}_{\text{CV+SS}}(i,n) = \sum_{\ell=1}^{L} W_i^{(\ell)} \widehat{Y}^{(\ell)}(i, N_i^{(\ell)}). \tag{20}$$

The variance of the stratified controlled-sum estimator is

$$\begin{aligned}
\text{Var}\left[\widehat{\theta}_{\text{CV+SS}}(i,n)\right] &= \sum_{\ell=1}^{L} \left(W_i^{(\ell)}\right)^2 \text{Var}\left[\widehat{Y}^{(\ell)}(i, N_i^{(\ell)})\right] \\
&= \sum_{\ell=1}^{L} \left(W_i^{(\ell)}\right)^2 \frac{\text{Var}\left[Y_{ij}^{(\ell)} - (\mathbf{C}_{ij}^{(\ell)} - \boldsymbol{\mu}_i^{(\ell)})^T \widehat{\boldsymbol{\beta}}_i(m_0)\right]}{N_i^{(\ell)}} \qquad (21) \\
&= \frac{1}{n} \sum_{\ell=1}^{L} W_i^{(\ell)} \text{Var}\left[Y_{ij}^{(\ell)} - (\mathbf{C}_{ij}^{(\ell)} - \boldsymbol{\mu}_i^{(\ell)})^T \widehat{\boldsymbol{\beta}}_i(m_0)\right] \qquad (22) \\
&= \frac{1}{n} \sum_{\ell=1}^{L} W_i^{(\ell)} \left(\frac{m_0 - 2}{m_0 - q_i - 2}\right) (1 - (R_i^{(\ell)})^2) \text{Var}[Y_{ij} | D_{ij} \in A_\ell]. \qquad (23)
\end{aligned}$$

Equation (21) holds because each observation $(Y_{ij}^{(\ell)}, \mathbf{C}_{ij}^{(\ell)})$ is i.i.d. within each stratum and $\widehat{\boldsymbol{\beta}}_i(m_0)$ is independent of them. We employ the proportional allocation scheme because it is easier to apply compared with the optimal allocation scheme. Therefore we have $N_i^{(\ell)} = n \times W_i^{(\ell)}$, which leads to Equation (22). Equation (23) follows from the assumed linear CV+SS combined model, with a similar derivation as in Tsai and Nelson [43]. For an easy comparison with other models, we would like to obtain a simplified representation of $\text{Var}[Y_{ij}|D_{ij} \in A_\ell]$. Glasserman et al. [15] showed that $\text{E}[Y_{ij}|D_{ij} \in A_\ell]$ converges to $\text{E}[Y_{ij}|D_{ij}]$ as we infinitely refine the stratification. In this case, we can apply a linear regression on Y and then write $Y_{ij} = \text{E}[Y_{ij}|D_{ij}] + \epsilon_{ij}$, where $\text{E}[Y_{ij}|D_{ij}]$ and ϵ_{ij} are uncorrelated (Glasserman [14]). If $\text{E}[Y_{ij}|D_{ij}]$ is a linear function of D_{ij}, then the variance after applying stratification will be further reduced as the number of strata L increases, and it will be equivalent to the residual variance when using D_{ij} as the control variable. Consequently, if we assume that $\{Y_{ij}, D_{ij}\}$ is bivariate normally distributed, we can have $\text{Var}[Y_{ij}|D_{ij} \in A_\ell] = \text{Var}[Y_{ij}] \times (1 - R_{Y,D}^2)$ (based on a similar derivation as in Model 0). We can see that the larger $R_{Y,D}^2$ is, the more variance reduction will be obtained when applying the CV+SS combined model. In addition, if we also assume that Model 0 holds and compare Equation (23) with the variance term of Equation (10), then we can obtain $\text{Var}\left[\widehat{\theta}_{\text{CV+SS}}(i,n)\right] \leq \text{Var}\left[\bar{Y}_i[m_0, m_0+n]\right]$ as long as the following condition is satisfied:

$$\sum_{\ell=1}^{L} W_i^{(\ell)} \left(1 - (R_i^{(\ell)})^2\right) \text{Var}[Y_{ij}|D_{ij} \in A_\ell] \leq \left(1 - R_{Y,\mathbf{C}}^2\right) \text{Var}[Y_{ij}]. \tag{24}$$

If we assume that $(R_i^{(\ell)})^2 = R^2_{Y,\mathbf{C}}, \forall \ell = 1, 2, \ldots, L$, then the inequality is valid for sure because the variance of the stratified sampling estimator (i.e., $\sum_{\ell=1}^{L} W_i^{(\ell)} \text{Var}[Y_{ij}|D_{ij} \in A_\ell]$) is smaller than or equal to the original output variance (i.e., $\text{Var}[Y_{ij}]$) (see Section 4.3 of Glasserman [14]). In a general case, we simply have to assume the independence between D_{ij} and $\mathbf{C}_{ij}$ to ensure that Inequality (24) holds (see Remark 4.1). In addition, it might be better to use the variable D_{ij} as a stratification variable instead of including it in the control vector $\mathbf{C}_{ij}$ because of the following reasons (i.e., the incremental benefit of using SS in a CV model might be more significant compared to another CV model with more control variables). First, in this way we do not need to suffer more from the loss ratio due to an increased value of q_i, as presented in Equation (8) and (23) for instance. Second, from the theoretical point of view, using the SS approach alone might reduce the variance more compared to simply using the CV approach because CV can only remove the variance associated with the linear part of $\text{E}[Y_{ij}|D_{ij}]$ (see p.220 of Glasserman [14] for an explanation).

Let $S_\ell^2(i, N_i^{(\ell)})$ represent the sample variance of the controlled responses of system i for those replications whose stratification variables fall in the ℓth stratum. Then, a natural estimator of $\text{Var}\left[\widehat{\theta}_{\text{CV+SS}}(i,n)\right]$ is as follows:

$$\widehat{\text{Var}}\left[\widehat{\theta}_{\text{CV+SS}}(i,n)\right] = \sum_{\ell=1}^{L} \left(W_i^{(\ell)}\right)^2 \frac{1}{N_i^{(\ell)}} S_\ell^2(i, N_i^{(\ell)}). \tag{25}$$

Fixed the number of strata L, we then obtain the asymptotic result as the total sample size n goes to infinity (combining the above results with those of Section 4.3.1 in Glasserman [14]):

$$\sqrt{n}\left(\widehat{\theta}_{\text{CV+SS}}(i,n) - \theta_i\right) \Rightarrow N(0, \xi_i^2), \tag{26}$$

where $\xi_i^2 = \sum_{\ell=1}^{L} W_i^{(\ell)} \text{Var}\left[Y_{ij}^{(\ell)} - (\mathbf{C}_{ij}^{(\ell)} - \boldsymbol{\mu}_i^{(\ell)})^T \widehat{\boldsymbol{\beta}}_i(m_0)\right]$ under the scheme of proportional allocation. Moreover, a consistent estimator of ξ_i^2 is $\sum_{\ell=1}^{L} W_i^{(\ell)} S_\ell^2(i, N_i^{(\ell)})$.

When using the SS approach, the conditional sampling process used to generate observations falling in each stratum and computing stratum probabilities requires additional effort and is sometimes prohibitive. Therefore, the PS technique is considered in which we do not sample conditionally from the defined stratum, but instead perform ordinary random sampling and then assign appropriate weights according to its falling stratum (i.e., $N_i^{(\ell)}$ is random and not decided in advance). An obvious disadvantage of this approach is that some strata may be empty (i.e., none of the n samples falls in the ith stratum). For the convenience of analysis, we assume that all strata are nonempty (i.e., $N_i^{(\ell)} > 0$ for all i and ℓ). The combined estimator $\widehat{\theta}_{\text{CV+PS}}(i,n)$ and its corresponding variance estimator $\widehat{\text{Var}}\left[\widehat{\theta}_{\text{CV+PS}}(i,n)\right]$ are formulated in the same way as in Equations (20) and (25), and both of them can be shown to be unbiased and independent of each other. Furthermore, we can

also obtain the convergence result (26) for $\widehat{\theta}_{\text{CV+PS}}(i,n)$ by applying the first-order central limit theory (Glynn and Szechtman [16]), based on the CV+SS combined model. In other words, in the large-sample limit we can conclude that the poststratified sampling controlled-sum estimator $\widehat{\theta}_{\text{CV+PS}}(i,n)$ is as efficient as the stratified controlled-sum estimator $\widehat{\theta}_{\text{CV+SS}}(i,n)$ with proportional stratification.

Remark 4.1. Difficulties may arise when the stratification variable D_{ij} is correlated with the control vector $\mathbf{C}_{ij}$, which could occur especially when D_{ij} is a component of $\mathbf{C}_{ij}$. We then have to expend some computational effort to obtain $\boldsymbol{\mu}_i^{(\ell)}$ and even worse, it is not clear whether $R_i^{(\ell)}$ is smaller or larger than $R_{Y,\mathbf{C}}$. If $R_i^{(\ell)}$ is strictly smaller than $R_{Y,\mathbf{C}}$, the overall variance may be increased when using the combined model. This issue can be resolved by requiring that the stratification variable D_{ij} be independent of the control $\mathbf{C}_{ij}$, which can be easily achieved in practice. For instance, in a queueing model we can let the service time be the control variable and the inter-arrival time be the stratification variable (assuming the independence between them). To present a simple illustration, we assume that $q_i = 1$ and that $\{Y_{ij}, C_{ij}, D_{ij}\}$ is multivariate normally distributed. Then, the partial correlation coefficient between Y_{ij} and C_{ij} (given a possible value of D_{ij}) can be written as follows: $R_i = \frac{R_{Y,C} - R_{Y,D}R_{C,D}}{\sqrt{(1-R_{Y,D}^2)(1-R_{C,D}^2)}}$ (see Section 2.5.3 of Anderson [1]). Therefore, if we assume that the stratification variable is independent of the control variable (i.e., $R_{C,D} = 0$), we can then obtain $R_i \geq R_{Y,C}$, which immediately leads to Inequality (24). In this case, we can also easily obtain $\mu_i^{(\ell)} = \mu_i$ without any computation.

4.2 Fully Sequential Procedure with CV+PS

In this subsection we present an FSP that can employ the combined estimator $\widehat{\theta}_{\text{CV+PS}}(i,n)$, which is the same as in Equation (20) with the notation CV+SS replaced by CV+PS, described in Section 4.1. Similar to the FSP with CV+PCE, in this procedure we update the variance estimator whenever a single new observation is obtained. It should be noticed that for each system i, we only have to recompute the sample variance $S_\ell^2(i, N_i^{(\ell)})$ for stratum ℓ, which the new D_{ij} observation belongs to. This implies that the required computational overhead to update the variance estimator is equivalent to that of the procedures presented in Section 2.1 and Appendix B.3. The variance estimator $\widehat{\text{Var}}\left[\widehat{\theta}_{\text{CV+PS}}(i,n)\right]$ (the same as in Equation (25) with the notation CV+SS replaced by CV+PS) is a strongly consistent estimator of $\text{Var}\left[\widehat{\theta}_{\text{CV+PS}}(i,n)\right]$, which is asymptotically equivalent to the variance of the stratified controlled-sum estimator under the scheme of proportional allocation.

Ideally, we need to choose the number of strata L that is large enough to achieve a significant variance reduction but also small enough relative to the total sample size n to avoid the occurrence of empty stratum. In the case when a stratum is empty (i.e., $N_i^{(\ell)} = 0$), Glasserman [14]

suggested to replace that stratum sample mean with zero but then the poststratified estimator will be biased. Cochran [10] proposed combining two or more strata (including that empty stratum) before computing the poststratified estimator. For a general use of the proposed procedure, we suggest employing the original CV estimator when there is at least one stratum with less than two observations for these r outputs (in order to compute the sample variance within each stratum), and otherwise, we suggest employing the combined estimator $\widehat{\theta}_{\mathrm{CV+PS}}(i,r)$ (with a fixed value of L) (see Remark 4.2). The following procedure description is based on the above assumption (i.e., $N_i^{(\ell)} \geq 2$ for all i and ℓ) regardless of whether the sample size is n_0 or r.

Procedure 2 (Fully Sequential Procedure with CV+PS)**.**

Step 0. Setup: Select confidence level $1/k < 1-\alpha < 1$, IZ parameter $\delta > 0$, the preliminary-stage sample size $m_0 > q+2$, the number of strata $L \geq 2$, and the first-stage sample size $n_0 \geq 2L$. Let $a = -\log\left[2\alpha/(k-1)\right]$.

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. For each system $i \in I$, generate $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0\}$ and then compute the estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ according to Equation (7). For each system $i \in I$, perform additional independent sampling to generate $\{(Y_{ij}, \mathbf{C}_{ij}, D_{ij}), j = m_0+1, m_0+2, \ldots, m_0+n_0\}$, resulting $N_i^{(\ell)}$ samples be within stratum A_ℓ, where $\sum_{\ell=1}^{L} N_i^{(\ell)} = n_0$. Let r be the observation counter. Set $r = n_0$.

Step 2. Update: For each system $i \in I$, compute $\widehat{\theta}_{\mathrm{CV+PS}}(i,r)$ and its corresponding variance estimator $\widehat{\mathrm{Var}}\left[\widehat{\theta}_{\mathrm{CV+PS}}(i,r)\right]$, according to Equations (20) and (25), respectively.

Step 3. Elimination:

For any pair of systems i and h in I, $i \neq h$, compute

$$\begin{aligned} \hat{\sigma}_{ih}^2(r) &= \widehat{\mathrm{Var}}\left[\widehat{\theta}_{\mathrm{CV+PS}}(i,r)\right] + \widehat{\mathrm{Var}}\left[\widehat{\theta}_{\mathrm{CV+PS}}(h,r)\right], \\ Z_{ih}(r) &= \widehat{\theta}_{\mathrm{CV+PS}}(i,r) - \widehat{\theta}_{\mathrm{CV+PS}}(h,r). \end{aligned}$$

Set $I^{\mathrm{old}} = I$. Let

$$I = I^{\mathrm{old}} \setminus \left\{ i \in I^{\mathrm{old}} : Z_{ih}(r) < \min\left\{0, -\frac{a\hat{\sigma}_{ih}^2(r)}{\delta} + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\mathrm{old}} \text{ and } h \neq i \right\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 4. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r+1$ and take the rth sample $\left(Y_{i(m_0+r)}, \mathbf{C}_{i(m_0+r)}, D_{i(m_0+r)}\right)$ from

system $i \in I$. For each system $i \in I$, if the rth sample of the stratification variable is from stratum A_ℓ (i.e., $D_{i(m_0+r)} \in A_\ell$), then update $N_i^{(\ell)} = N_i^{(\ell)} + 1$, and also update $\widehat{Y}^{(\ell)}(i, N_i^{(\ell)})$ and $S_\ell^2(i, N_i^{(\ell)})$. Then go to Step 2.

Remark 4.2. For each system i, we suggest using the controlled-sum estimator $\bar{Y}_i[m_0, m_0 + r]$ and its corresponding variance estimator when there is at least one stratum with less than two observations for these r collected observations. Otherwise, we use the combined estimator $\widehat{\theta}_{\text{CV+PS}}(i, r)$ and its variance estimator $\widehat{\text{Var}}\left[\widehat{\theta}_{\text{CV+PS}}(i, r)\right]$. This can be implemented easily in our procedure because in **Step 3** we employ the summation of the marginal variance estimator for each pair of systems (instead of using the variance of the paired observations). The asymptotic statistical validity still holds under this setting.

We close this section by formally stating the asymptotic statistical validity of Procedure 2 as the second theorem.

Theorem 2. *Let* $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0\}$ *be the* m_0 *preliminary-stage outputs and control variables from system* i, $i = 1, 2, \ldots, k$, *yielding the estimator* $\widehat{\boldsymbol{\beta}}_i(m_0)$ *in Equation (7). Let* $\{(Y_{ij}, \mathbf{C}_{ij}, D_{ij}), j = m_0 + 1, m_0 + 2, \ldots, m_0 + n_0\}$ *be the* n_0 *first-stage outputs, control variables and stratification variables from system* i, *resulting* $N_i^{(\ell)}$ *samples be within stratum* A_ℓ, *where* $\sum_{\ell=1}^{L} N_i^{(\ell)} = n_0$ *and* L *is the number of strata. We assume that* $N_i^{(\ell)} \geq 2$ *for all* i *and* ℓ. *Furthermore, we assume that samples* $(Y_{ij}, \mathbf{C}_{ij}, D_{ij})$ *taken from the same system* i *are i.i.d. for* $j = 1, 2, \ldots$, *and also independent from those from any other system* $h \neq i$. *Let* θ_i *denote the unknown mean of system* i. *Without loss of generality, we let* $\theta_1 - \delta \geq \theta_2 \geq \ldots \geq \theta_k$, *where* δ *is the IZ parameter. Let the first-stage sample size* $n_0 = n_0(\delta)$ *be a function of* δ *such that* $n_0 \to \infty$ *and* $\delta^2 n_0 \to 0$ *as* $\delta \to 0$. *Then, as* $\delta \to 0$, *Procedure 2 selects system* 1 *as the best with a probability at least* $1 - \alpha$.

Remark 4.3. Under the additional assumption that there is no empty stratum, i.e., $N_i^{(\ell)} \geq 2$ for all i and ℓ, Theorem 2 can be considered as a special case of Theorem 1 without any U_{ij} but all $V_{i\ell}$ being applied with the CV+PS technique. In fact, this assumption can be easily satisfied in the asymptotic regime $\delta \to 0$ since the first-stage sample size $n_0(\delta) \to \infty$ implying that $N_i^{(\ell)} \to \infty$ for each $\ell = 1, 2, \ldots, L$ using the proportional allocation scheme.

5 Numerical Experiments

In this section, we conduct an extensive empirical study to compare our proposed procedures (with integrated VRTs) to different existing fully sequential procedures. We implement the ordinary fully sequential procedures of Kim and Nelson [21] (denoted as $\mathcal{KN}$) and Tsai and Nelson [43] (which

assumes Model 0 and is denoted as $\mathcal{TN}$) as well as their variance-updating versions (denoted as $\mathcal{KN}$-$\mathcal{U}$ and $\mathcal{TN}$-$\mathcal{U}$, respectively), which then serve as reasonable benchmark procedures as described in detail in Appendices B.2 and B.3. It is easy to show the asymptotic validity of these two procedures, which is a similar but simpler process than the proof of Theorem 1 shown in Appendix A.

The system outputs are represented as various configurations of k normal distributions. In all cases, system 1 has the largest true mean and is the best system. For the sake of simplicity, in the following description, we skip the notation of the replication counter j. Let Y_i be a simulation output from system i for $i = 1, 2, \ldots, k$. For simplicity, we assume that each system has one control variable (i.e., $q_i = 1$ for all i). We then assume Model 0 holds and that the observation can be represented as

$$Y_i = \theta_i + (C_i - \mu_i)\beta_i + \epsilon_i$$

where $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_\epsilon^2)$ random variables. The input random variables $\{C_i,\ i = 1, 2, \ldots, k\}$ are $\mathrm{N}(0, \sigma_c^2)$ random variables (i.e., $\mu_i = 0$) and independent of $\{\epsilon_i,\ i = 1, 2, \ldots, k\}$. We also set $\beta_i = 1$ for each system $i = 1, 2, \ldots, k$. Therefore, $\{Y_i,\ i = 1, 2, \ldots, k\}$ are distributed as $\mathrm{N}(\theta_i, \sigma_y^2)$ random variables, where $\sigma_y^2 = \sigma_c^2 + \sigma_\epsilon^2$. The squared correlation coefficient between Y_i and C_i is $R_{Y,C}^2 = \sigma_c^2/(\sigma_c^2 + \sigma_\epsilon^2)$ for each system $i = 1, 2, \ldots, k$. We set $\sigma_y^2 = 1$ and therefore $\sigma_c^2 = R_{Y,C}^2$ for each system $i = 1, 2, \ldots, k$.

For the CV+CE combined models, we let X_i denote the conditional variable used for system i. Further, in the Combined Model I and II we let $\{Y_i, X_i\}$ be bivariate normally distributed with a squared correlation $R_{Y,X}^2$. In the Combined Model III, we assume that $\{X_i, C_i\}$ is bivariate normally distributed with a squared correlation $R_{X,C}^2$. For the Combined Model II, we do not assume the value of $\beta_i^{(2)}$ and instead obtain its value by solving the following equation: $R_{\mathrm{E}[Y|X],C}^2 = R_{X,C}^2 = \frac{(\beta_i^{(2)})^2 \times \mathrm{Var}[C_i]}{\mathrm{Var}[\mathrm{E}[Y_i|X_i]]}$, where we know $\mathrm{Var}[\mathrm{E}[Y_i|X_i]] = R_{Y,X}^2$ (because $\sigma_y^2 = 1$) and $\mathrm{Var}[C_i] = R_{Y,C}^2$ (derived from Model 0). We can then obtain $\mathrm{Var}[\epsilon_i^{(2)}]$ using the following equation: $\mathrm{Var}[\mathrm{E}[Y_i|X_i]] = (\beta_i^{(2)})^2 \times \mathrm{Var}[C_i] + \mathrm{Var}[\epsilon_i^{(2)}]$. It should be noticed that the choice of $R_{Y,C}^2$ is irrelevant to the procedure performance because the value of $\mathrm{Var}[\epsilon_i^{(2)}]$ remains the same as long as $R_{Y,X}^2$ and $R_{X,C}^2$ are determined. A similar variable generation process is implemented for the Combined Model I and III. However, in the simulation experiments of Combined Model I, it is also necessary to assign a parameter ρ such that $0 < \rho < 1$ and $\mathrm{Var}[\mathrm{E}[C_i|X_i]] = \mathrm{Var}[C_i] \times \rho$. Following this setting, we can have $R_{X,\mathrm{E}[C|X]}^2 = R_{X,C}^2 \times (\frac{1}{\rho})$. Notice that in Combined Model I we do not simultaneously assume that $\{Y_i, X_i\}$ and $\{C_i, X_i\}$ are bivariate normally distributed which will result in an extreme case where a 100% variance reduction can be achieved.

For the CV+PS combined model, we let D_i denote the stratification variable used for system i, which is assumed to be normally distributed and correlated with Y_i (with a squared correlation

$R^2_{Y,D}$), but is independent from C_i (i.e., $R^2_{C,D} = 0$). When applying the stratified sampling technique, we must know the exact distribution of the stratification variable, which is usually assumed to be normally distributed, at least asymptotically, in the related literature. For the determination of the boundaries between the strata, the experimental results from existing literature often reveal that Sethi's stratification scheme is better than the equal-probability scheme (e.g., Sabuncuoglu et al. [39]). Therefore, in the current experiments we use Sethi's optimal stratification scheme for a normal random variable under proportional allocation to classify each of the replicated observations $\{Y_i, C_i, D_i\}$ into the appropriate stratum. In the literature on PS technique, L is often specified in the range of $2 \leq L \leq 6$, where it is experimentally found that the marginal efficiency gain may not be significant when using more than 4 strata (see Chapter 5 of Cochran [10], Sabuncuoglu et al. [39], and Wilson and Pritsker [46]). In the following experimental study we use $L = 2$ and 4.

We compare the performance of each FSP on different configurations of the systems, with examining factors including the practically significant difference δ, the number of systems k, the number of strata L (when the PS technique is used), the configurations of the system means θ_i, and the squared correlation coefficients $R^2_{Y,C}$, $R^2_{Y,X}$, $R^2_{X,C}$, and $R^2_{Y,D}$. Common random numbers are not employed. The configurations, the experimental design, and the results are described below.

5.1 Configurations and Experimental Design

We investigate the slippage configuration (SC) of the true means of the systems in which θ_1 is set to exactly δ, while $\theta_2 = \theta_3 = \cdots = \theta_k = 0$. This is the most difficult scenario in which to achieve the pre-specified PCS, because all the inferior systems are very close to the best system. We choose $\delta = \sqrt{(\sigma_c^2 + \sigma_\epsilon^2)/n_0}$; therefore, the indifference-zone parameter can be interpreted as one standard deviation of the first-stage sample mean. To examine the efficiency of these procedures in eliminating inferior systems, the configuration of monotone-decreasing means (MDM) is also used. In the MDM configuration, the means of systems are determined according to the following formula: $\theta_1 = \delta$ and $\theta_i = \theta_1 - (i-1)(\delta/2)$, for $i = 2, 3, \ldots, k$.

The number of systems simulated in each experiment is varied with $k = 10, 30, 50, 100$. In all experiments, we set the nominal PCS $1 - \alpha = 0.95$, the preliminary-stage sample size $m_0 = 10$, and the first-stage sample size $n_0 = 20$. These algorithm parameter settings are based on the guidelines provided in Tsai and Kuo [42] and Tsai and Nelson [43]. For each configuration, 500 trials of each procedure are performed to compare the performance measures, including the estimated PCS and the average number of simulated observations per system (ANS). To simplify the presentation, we round the values of PCS and ANS to the nearest hundredth and integer number, respectively.

5.2 Summary of the Results

The PCS of the proposed fully sequential procedures using the combined models (either with or without the variance updating mechanism) is higher than the nominal level 0.95 in most configurations. We also observe that the required correlation threshold for our combined models to improve on the traditional models is not difficult to achieve. Instead of presenting comprehensive results from such a large simulation study, we demonstrate details of some specific results that highlight the key conclusions.

5.2.1 Fully Sequential Procedures with CV+PCE

In Table 3, we evaluate the effect of different levels of various correlations on the performance of the FSP with CV+PCE (which is described in Section 2.1) where Model 0 and Combined Model II are used as a combination. We consider the slippage configuration and let P_{CE} denote the probability that there exists a conditional variable X_i such that $\text{E}[Y_i|X_i]$ can be obtained (i.e., Combined Model II can be applied) in a single simulation replication. In other words, Model 0 is employed in each replication with a probability $1 - P_{\text{CE}}$. The PCS of the FSP with CV+PCE is still greater than or equal to the nominal level 0.95, but it is not so conservative as the previously presented $\mathcal{TN}$-like procedure with CV+CE (i.e., its PCS is closer to 0.95). As shown in Table 3, a larger P_{CE} value makes the procedure more efficient in terms of reduced ANS because a greater efficiency gain can be reaped from the combined model. It should also be noticed that the CV+PCE procedure should be equivalent to the CV+CE procedure with Combined Model II when $P_{\text{CE}} = 1$. However, the ANS values when $P_{\text{CE}} = 1$ are smaller than those presented in Table 9 because in the CV+PCE procedure, the variance estimator is updated sequentially. Thus, the required consumption of ANS is smaller, and we can only achieve an asymptotic PCS guarantee.

Note that when implementing the CV+PCE scheme in the numerical experiments, we actually use the following equation (without loss of generality, we take Combined Model II as an example),

$$\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n) = \bar{Y}^{(2)}_i(n) - \left(\bar{\mathbf{C}}^{(2)}_i(n) - \boldsymbol{\mu}_i\right)^T \widehat{\boldsymbol{\beta}}_i(m_0)$$

where $\widehat{\boldsymbol{\beta}}_i(m_0)$ is obtained from the pure CV model with m_0 independent samples in a preliminary stage. In other words, we use the same $\widehat{\boldsymbol{\beta}}_i(m_0)$ for both $\widehat{\theta}_{\text{CV}}(i,n)$ and $\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n)$. This compromise is due to the fact that there might be a very small number (or none) of CV+CE observations in the preliminary stage for some systems. Based on this setting, we then assume that $\boldsymbol{\beta}_i = \boldsymbol{\beta}^{(2)}_i$ (which implies that $\text{Cov}\left[\bar{Y}^{(2)}_i(n), \bar{\mathbf{C}}^{(2)}_i(n)\right] = \text{Cov}\left[\bar{Y}_i(n), \bar{\mathbf{C}}_i(n)\right]$) to obtain the following more convenient result. In order to make sure that the additional benefit of reducing variances by CV+CE is achieved (i.e., $\text{Var}\left[\widehat{\theta}^{(2)}_{\text{CV+CE}}(i,n)\right] \leq \text{Var}\left[\widehat{\theta}_{\text{CV}}(i,n)\right]$), we only need to require that

Table 3: Performance measures for our CV+PCE fully sequential procedures under the SC when $m_0 = 10$, $n_0 = 20$, $q = 1$ and $1 - \alpha = 0.95$.

P_{CE}	$R^2_{Y,C}$	$R^2_{Y,X}$	$R^2_{X,C}$	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
				PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
0	0.5	0.6	0.4	0.96	74	0.95	89	0.97	97	0.97	98
0.2	0.5	0.6	0.4	0.96	65	0.96	79	0.97	89	0.97	89
0.5	0.5	0.6	0.4	0.95	55	0.96	64	0.97	72	0.97	74
0.8	0.5	0.6	0.4	0.96	45	0.97	51	0.95	55	0.97	58
1	0.5	0.6	0.4	0.95	39	0.95	45	0.96	46	0.97	49
0	0.5	0.5	0.5	0.96	74	0.95	89	0.97	97	0.97	98
0.2	0.5	0.5	0.5	0.95	64	0.96	76	0.97	86	0.95	87
0.5	0.5	0.5	0.5	0.96	49	0.95	56	0.97	65	0.96	65
0.8	0.5	0.5	0.5	0.96	37	0.98	40	0.96	44	0.97	44
1	0.5	0.5	0.5	0.99	32	0.96	33	0.95	34	0.95	35
0	0.5	0.4	0.6	0.96	74	0.95	89	0.97	97	0.97	98
0.2	0.5	0.4	0.6	0.96	61	0.96	74	0.98	83	0.99	84
0.5	0.5	0.4	0.6	0.95	45	0.95	52	0.96	56	0.95	57
0.8	0.5	0.4	0.6	0.95	34	0.96	35	0.95	37	0.95	37
1	0.5	0.4	0.6	0.98	30	0.99	30	0.97	30	0.96	30

$\text{Var}\left[\epsilon^{(2)}_{ij}\right] \leq \text{Var}\left[\epsilon_{ij}\right]$, which is equivalent to the condition (17).

5.2.2 Fully Sequential Procedures with CV+PS

In the experiments on fully sequential procedures with CV+PS, we compare the procedure presented in Section 4.2 to $\mathcal{KN}$-$\mathcal{U}$ and $\mathcal{TN}$-$\mathcal{U}$ under the slippage configuration with the settings of $L = 2$ and 4 (see Table 4 and 5). The CV+PS fully sequential procedure can still achieve the pre-specified PCS guarantee (which can only be proven asymptotically) and yields better performance in terms of ANS as anticipated when the value of $R^2_{Y,C}$ or $R^2_{Y,D}$ is increased. The experimental results also indicate that the CV+PS procedure is superior to $\mathcal{TN}$-$\mathcal{U}$ when the value of $R^2_{Y,D}$ is greater than or equal to 0.1, which shows that the required correlation threshold is easier to achieve as compared to the case of CV. We find that the advantage of our CV+PS procedure relative to $\mathcal{KN}$-$\mathcal{U}$ and $\mathcal{TN}$-$\mathcal{U}$ in terms of ANS is more significant when $L = 4$, which is due to the fact that more stratification layers naturally lead to more variance reduction. In the current experiments with $k = 100$, the additional number of replications (for each system on average) required to be able to apply PS after the first stage is 2.04 and 17.73 for the cases of $L = 4$ and $L = 6$, respectively. In the case of $L = 2$, there is no empty stratum after the first-stage sampling process in all examined cases.

Table 4: Performance measures for our CV+PS fully sequential procedures in comparison with $\mathcal{KN}$-$\mathcal{U}$ and $\mathcal{TN}$-$\mathcal{U}$ under the SC when $L = 2$, $m_0 = 10$, $n_0 = 20$, $q = 1$ and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	$R^2_{Y,D}$	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$-$\mathcal{U}$			0.98	116	0.95	141	0.99	151	0.96	169
$\mathcal{TN}$-$\mathcal{U}$	0.3		0.96	105	0.95	126	0.97	134	0.95	146
CV+PS	0.3	0.1	0.96	93	0.98	121	0.97	126	0.97	142
	0.3	0.3	0.97	80	0.99	105	0.98	105	0.98	119
	0.3	0.5	0.96	64	0.97	85	0.96	84	0.98	96
	0.3	0.7	0.95	50	0.98	66	0.95	65	0.98	73
$\mathcal{TN}$-$\mathcal{U}$	0.5		0.96	74	0.95	92	0.97	99	0.97	102
CV+PS	0.5	0.1	0.96	69	0.97	87	0.98	95	0.98	97
	0.5	0.3	0.95	54	0.96	70	0.99	75	0.99	75
$\mathcal{TN}$-$\mathcal{U}$	0.7		0.96	51	0.95	56	0.97	60	0.95	62
CV+PS	0.7	0.1	0.96	46	0.95	53	0.98	58	0.97	56
	0.7	0.3	0.96	36	0.96	38	0.96	40	0.95	38

Table 5: Performance measures for our CV+PS fully sequential procedures in comparison with $\mathcal{KN}$-$\mathcal{U}$ and $\mathcal{TN}$-$\mathcal{U}$ under the SC when $L = 4$, $m_0 = 10$, $n_0 = 20$, $q = 1$ and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	$R^2_{Y,D}$	$k = 10$		$k = 30$		$k = 50$		$k = 100$	
			PCS	ANS	PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$-$\mathcal{U}$			0.98	116	0.95	141	0.99	151	0.96	169
$\mathcal{TN}$-$\mathcal{U}$	0.3		0.96	105	0.95	126	0.97	134	0.95	146
CV+PS	0.3	0.1	0.95	93	0.99	119	0.98	120	0.97	128
	0.3	0.3	0.95	72	0.96	86	0.95	90	0.98	96
	0.3	0.5	0.96	52	0.98	59	0.97	61	0.95	64
	0.3	0.7	0.97	36	0.95	40	0.97	40	0.96	42
$\mathcal{TN}$-$\mathcal{U}$	0.5		0.96	74	0.95	92	0.97	99	0.97	102
CV+PS	0.5	0.1	0.95	67	0.96	78	0.96	80	0.96	89
	0.5	0.3	0.96	47	0.95	52	0.97	55	0.99	58
$\mathcal{TN}$-$\mathcal{U}$	0.7		0.96	51	0.95	56	0.97	60	0.95	62
CV+PS	0.7	0.1	0.95	45	0.97	46	0.96	48	0.98	50
	0.7	0.3	0.96	31	0.98	32	0.96	32	0.95	32

6 Illustrative Examples

In this section we first consider the Markovian queue with state-dependent service (e.g., refer to Section 2.9 in Gross et al. [18]). We assume that the system has Poisson arrivals, exponentially distributed service times, one server, an infinite buffer, and a first-come-first-served queueing discipline. The customer arrival rate λ_i is constant but different for each system i. We want to allow the server to become faster when there are more customers waiting in queue, which is referred as the dynamic service rate control policy in the literature (e.g., Kumar et al. [25]). The service rate for customer m from system i is specified as $\mu_i + 0.01 \times (Q_m - 6)$, where Q_m is random and represents the number of customers waiting in line when the mth customer just enters the server. Each procedure is implemented for ten different configurations, where the performance measure is the steady-state mean of the waiting time in the system. The ten configurations along with their true expected waiting times, which are computed analytically, are given in Table 6 (see Section 2.9 in Gross et al. [18] for the derivation). Notice that a system with a smaller waiting time is better, so System 1 consisting of one fastest server is the best system (but does not have the lowest arrival rate).

All candidate systems are initialized in a steady state to mitigate the initial transient bias. That is, for each replication we sample the initial condition in accordance with the steady-state distribution of the number of customers in the system, which can be computed in advance. For each replication, we use the average waiting time for thirty customers as the output. For the procedures involving control variates, the average service time for thirty customers is used as the control variable for the replication j, which means that

$$Y_{ij} = \frac{\sum_{m=1}^{30} W_{ijm}}{30} \quad \text{and} \quad C_{ij} = \frac{\sum_{m=1}^{30} S_{ijm}}{30},$$

where W_{ijm} is the waiting time in the system for customer m of replication j from system i, and S_{ijm} is the service time for customer m of replication j from system i. For the fully sequential selection procedures that employ sample means (Kim and Nelson [21]), we choose the first-stage sample size $n_0 = 30$. For all the CV-related procedures, the preliminary-stage and first-stage sample size are set to $m_0 = 10$ and $n_0 = 20$. The nominal probability of correct selection is set to $1 - \alpha = 0.95$. The indifference-zone parameter is set to $\delta = 0.1$, and thus the correct selection is to choose System 1 because the difference between the expected waiting times of System 1 and 2 is greater than 0.1.

6.1 Results of the CV+CE and CV+PS Procedures

Suppose that there are p customers either waiting in line or in service when the mth customer arrives. When CV+CE Combined Model II is applied, we use the following conditional expectation of outputs: $\mathrm{E}[W_{ijm}|Q_{m-1}, Q_{m-2}, \ldots, Q_{m-p}] = \sum_{r=m-p}^{m-1} (\mu_i + 0.01 \times (Q_r - 6))^{-1}$. The conditional expectation equals to zero when $p = 0$. When CV+CE Combined Model III is applied, we then use the following conditional expectation of controls: $\mathrm{E}[S_{ijm}|Q_m] = (\mu_i + 0.01 \times (Q_m - 6))^{-1}$. It should be noted that CV+CE Combined Model I cannot be applied in this case because the conditional variables for W_{ijm} and S_{ijm} are different. When applying the PS technique, we choose the inter-arrival time as the stratification variable (also taking the average of thirty customers) and specify $L = 2$. It should also be noted that the control variable and the stratification variable are independent from each other. We use Sethi's optimal stratification scheme to determine the boundaries between the strata, and use proportional allocation scheme to decide the allocation fractions.

The following experimental results can be explained by the estimated correlations based on 1000 replications (for System 1): $R^2_{Y,C} = 0.272$, $R^2_{Y,A} = 0.153$ and $R^2_{Y,Q} = 0.003$, where A and Q represent the average inter-arrival time and the average number of waiting customers for the first thirty customers in each replication. Table 7 presents the results of the procedures developed in this paper and compares them to those of other procedures with 500 complete macro-replications. We also test the FSP of Tsai and Nelson [43] where service times and inter-arrival times are both used as the control variables (denoted as $\mathcal{TN}$-2). The observed PCS of all procedures are greater than the nominal level 0.95. In terms of ANS, it is not surprising that $\mathcal{TN}$ and $\mathcal{TN}$-$\mathcal{U}$ are superior to $\mathcal{KN}$ and $\mathcal{KN}$-$\mathcal{U}$, respectively (because $R^2_{Y,C}$ is greater than 0.2). We can also see that $\mathcal{TN}$-2 is better than $\mathcal{TN}$ which implies that, in this case, the inclusion of one more control variable can increase the correlation between the output and controls, and the gains more than offset the losses incurred by the loss ratio. The use of CV+CE Combined Model II can result in a more significant improvement; we can see that there is a reduction of 21% in the ANS when compared to $\mathcal{TN}$. However, the use of CV+CE Combined Model III results in a worse performance when compared to $\mathcal{TN}$, which is an anticipated result because $R^2_{Y,Q}$ could be as low as 0.003 (which also follows from the previous analytical results discussed in Section 3.2.3). The CV+PS procedure also leads to a reduction in ANS when compared to $\mathcal{TN}$-$\mathcal{U}$ (because $R^2_{Y,A}$ could be greater than 0.1).

6.2 Results of the CV+PCE Procedure

To illustrate the applicability of the proposed CV+PCE procedure, an additional feature of multiclass customers is added to the original queueing system (see Chapter 5 of Gautam [13] for

Table 6: The ten queueing systems with dynamic service rate ($= \mu_i + 0.01 \times (Q_m - 6)$) and their expected waiting times in steady state.

System i	λ_i	μ_i	$\mathrm{E}[W_i]$
1	4	5.00	0.964
2	3	3.90	1.090
3	2	2.90	1.119
4	4	4.85	1.105
5	3	3.85	1.148
6	2	2.85	1.182
7	4	4.80	1.159
8	3	3.80	1.211
9	2	2.80	1.252
10	4	4.75	1.218

an introduction of multiclass queueing systems). We assume that there are two classes of customers with different service time distributions (exponentially and normally distributed). The variance of the normal distribution is assumed to be equal to the square of its mean. The dynamic service rates of both types of customers are specified in the same way as in the previously described Markovian model. Suppose that there are $p > 0$ customers either waiting in line or in service when the mth customer arrives. We can then obtain $\mathrm{E}[W_{ijm}|Q_{m-1}, Q_{m-2}, \ldots, Q_{m-p}] = \sum_{r=m-(p-1)}^{m-1} (\mu_i + 0.01 \times (Q_r - 6))^{-1} + \mathrm{E}[S_{ij(m-p)} - s | S_{ij(m-p)} > s]$, where the notation s represents the amount of time the $(m-p)$th customer already spent in service when the mth customer arrives. It is difficult to compute $\mathrm{E}[S_{ij(m-p)} - s | S_{ij(m-p)} > s]$ when $S_{ij(m-p)}$ comes from the assumed normal distribution, and therefore it is replaced with the actual remaining service time. More specifically, if all of the thirty customer waiting times (i.e., $\{W_{ijm}, \forall m = 1, 2, \ldots, 30\}$) can be applied with the CE technique (i.e., $S_{ij(m-p)}$ comes from the exponential distribution for the thirty customers), so for that particular replication we can then employ the estimator V_{ij} defined in the CV+PCE procedure of Section 2.1. On the other hand, if any of the thirty customer waiting times cannot be applied with CE, we then use the estimator U_{ij} for the jth replication. In the experiment, we specify that an arriving customer with normally distributed service time occurs with a probability of 0.025. Table 8 presents the results of the CV+PCE procedure in comparison with other procedures in 500 complete macro-replications. The results show a significant improvement in terms of ANS when using the CV+PCE procedure.

Table 7: Results for our procedures with combined models in comparison with the existing procedures in 500 Trials with $\delta = 0.1$ and $1 - \alpha = 0.95$.

Procedure	PCS	ANS	Procedure	PCS	ANS
$\mathcal{KN}$	0.988	202	$\mathcal{KN}$-$\mathcal{U}$	0.984	186
$\mathcal{TN}$	0.996	189	$\mathcal{TN}$-$\mathcal{U}$	0.982	169
$\mathcal{TN}$-2	0.992	183	CV+PS	0.978	156
CV+CE-II	0.990	150	CV+CE-III	0.994	231

Table 8: Results for our CV+PCE procedure in comparison with the existing procedures in 500 Trials with $\delta = 0.1$ and $1 - \alpha = 0.95$ (with two classes of customers).

Procedure	PCS	ANS
$\mathcal{KN}$	0.990	207
$\mathcal{TN}$	0.996	196
$\mathcal{KN}$-$\mathcal{U}$	0.990	171
$\mathcal{TN}$-$\mathcal{U}$	0.988	161
CV+PCE	0.990	142

7 Conclusions

In this paper, we propose specific combined models and corresponding fully sequential selection procedures to jointly employ CV, CE and PS techniques. We also compare them to ordinary fully sequential procedures that use sample means or pure CV estimators via statistical analysis and experimental study. It should be noted that most of the existing works only evaluated the performance of VRTs based on simulation experiments for specific stochastic systems. For the case of jointly applying CV and CE, based on analytical results we demonstrate that Combined Model I is better than Combined Models II and III, and that both Combined Models I and II are superior to the pure CV model. We have not found comparative studies among these three combined models in any existing literature that we know. More specifically, applying CE to the output and control variables simultaneously is more beneficial than only applying CE to the output, but applying CE alone to the controls is not helpful beyond the given CV model. The CV+CE combined models can deliver better efficiency when the conditional variable is less correlated with the output and is significantly correlated with the control variables (although we may not have much choice). By contrast, when applying the CV+PS combined model, we require that the exact distribution of the stratification variable is known, and in the meantime prefer to choosing a stratification variable that is largely correlated with the output but less correlated with the control variables. It should be noted that the proposed CV+PS estimator is new and appropriate for use in fully sequential

procedures. The marginal benefit of using CE or PS could be greater than that of using CV because they do not suffer from the loss ratio (see Section 3.1). We have presented fully sequential selection procedures that can employ CV+CE combined models and can be shown to be statistically valid with finite samples. We also propose fully sequential procedures for CV+PCE and CV+PS combined estimators whose statistical validity can be proven in the asymptotic regime. In the proposed procedures, the application of the CV+CE or CV+PS models would not increase the computational overhead much beyond that of the CV model.

There are several possible directions to extend or improve the developed approaches. For instance, it has been shown that both CE and PS are particularly effective in rare event problems where excessive simulation time is generally required to collect a sufficient number of observations (see Lavenberg and Welch [27] and Rubino and Tuffin [38] for illustrative examples where these techniques are applied). Therefore an extensive experimental evaluation of rare event problems using the proposed VRT combined schemes would be a worthwhile follow-up research work. Additional efficiency improvement of the proposed procedures may be possible if we employ a CV+SS combined model. However, in this case, it is not clear how to determine which stratum the next generated observation should belong to (i.e., the sequence of stratified samples). Notice that in the literature, the SS technique is often implemented by taking batches; however, with the setting in our target problem, a fully sequential procedure is expected to become inefficient if a "stage" is defined by a batch mean, and a large batch size is often required as well. The other possible topic of future study is to consider VRT combined models where the control mean is unknown and must be estimated (Pasupathy et al. [33]).

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A Proof of Theorem 1

The proof of Theorem 1 follows similar argument as proving the statistical validity of the APS (Asymptotic Parallel Selection) Procedure in Luo et al. [29], which also need to establish the following important lemmas. It is worthwhile pointing out that constructing the Brownian motion process in this paper is different from that in Luo et al. [29].

We start from considering the slippage configuration where the difference between the mean of the best and all other systems equals to the IZ parameter δ, i.e., $\theta_2 = \theta_3 = \cdots = \theta_k = \theta_1 - \delta$, and then demonstrate the validity of general IZ settings in the finial proof.

Consider any pair of systems, systems 1 and h, where $h = 2, 3, \ldots, k$. Define

$$N_{1h}^{\delta} = \lceil 2a(p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2)/\delta^2 \rceil,$$

where $a = -\log[2\alpha/(k-1)]$ and $1 - q_i = p_i = \lim_{r\to\infty} m_{ir}/r$, $i = 1$ or h, is limiting proportion as specified in Theorem 1, and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then N_{1h}^{δ} is the maximum number of observations needed to make the elimination decision of either system 1 or h. Let $s = r/N_{1h}^{\delta}$ be any number in $[0, 1]$. Define the stochastic process $\widetilde{Z}_{1h}(s)$,

$$\begin{aligned}
\widetilde{Z}_{1h}(s) &= \begin{cases} \frac{1}{\sqrt{p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2}} \cdot \frac{r}{\sqrt{N_{1h}^{\delta}}} Z_{1h}(r), & n_0 \le r \le N_{1h}^{\delta}, \\ 0, & 0 \le r < n_0, \end{cases} \\
&= \begin{cases} \frac{1}{\sqrt{p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2}} \cdot \frac{U_1(m_{1r}) + V_1(n_{1r}) - U_h(m_{hr}) - V_h(n_{hr})}{\sqrt{N_{1h}^{\delta}}}, & \frac{n_0}{N_{1h}^{\delta}} \le s \le 1, \\ 0, & 0 \le s < \frac{n_0}{N_{1h}^{\delta}}, \end{cases}
\end{aligned} \tag{27}$$

where $Z_{1h}(r)$ is defined in Equation (6). In fact, we care only about the process $\widetilde{Z}_{1h}(s)$ evolves in the interval $s \in [n_0/N_{1h}^{\delta}, 1]$ since the pairwise comparisons are conducted only when $r \ge n_0$. However, for mathematical rigorousness, we force $\widetilde{Z}_{1h}(s) = 0$ when $s \in [0, n_0/N_{1h}^{\delta})$, which is well defined as shown in the following lemma.

Lemma 1 (Convergence to a Brownian Motion Process)**.** *Let* $\mathbf{D}[0, 1]$ *be the Skorohod space of all right-continuous real-valued functions on* $[0, 1]$ *with limits from the left everywhere, endowed with the Skorohod* J_1 *topology. Thus,* $Z_{1h}(\cdot)$ *defined by Equation* (27) *with* $h = 2, 3, \ldots, k$ *is an element of the Skorohod space* $\mathbf{D}[0, 1]$*. Suppose that the conditions in Theorem 1 are all satisfied. Then, under the SC, i.e.,* $\theta_2 = \theta_3 = \cdots = \theta_k = \theta_1 - \delta$*, we have*

$$\widetilde{Z}_{1h}(\cdot) \Rightarrow \mathcal{B}_{\Delta}(\cdot), \quad \text{as } \delta \to 0,$$

where $\mathcal{B}_{\Delta}(s) = \mathcal{B}(s) + \Delta s$*, a standard Brownian motion process with a constant drift* $\Delta = \sqrt{2a}$*.*

Proof. Let $\epsilon^\delta = n_0/N_{1h}^\delta$. Recall that $n_0 \to \infty$ and $\delta^2 n_0 \to 0$ as $\delta \to 0$. Then, $N_{1h}^\delta \to \infty$ and $\epsilon^\delta \to 0$ as $\delta \to 0$. Notice that $\widetilde{Z}_{1h}(s) = 0$ for $0 \le s < \epsilon^\delta$, which implies that

$$\widetilde{Z}_{1h}(0) = \mathcal{B}_\Delta(0) = 0 \text{ and } \widetilde{Z}_{1h}(\cdot) \text{ is right-continuous at } s = 0 \text{ for all } \delta. \tag{28}$$

We next focus only on $s \in [\epsilon^\delta, 1]$, that is, $r \in [n_0, N_{1h}^\delta]$.

We first analyze the term $\frac{U_i(m_{ir})}{\sqrt{N_{1h}^\delta}}$, $i = 1$ or h, on the right-hand-side (RHS) of Equation (27). Note that the same argument can be applied on $\frac{V_i(n_{ir})}{\sqrt{N_{1h}^\delta}}$. At stage r, let $a_{ir} = \frac{m_{ir}}{r}$, which is random variable converges to p_i with probability 1 (w.p.1). We mainly focus on the nontrivial case that $p_i \in (0,1)$. For the case that $p_i = 0$ or 1, the derivation is relatively easy and can be incorporated in the same formula. Let $b_{ir} = a_{ir} N_{1h}^\delta$. Then, as $\delta \to 0$, by Theorem 14.4 in Billingsley [4] and Theorem 11.4.5 in Whitt [45], we know that

$$\begin{aligned} \frac{U_i(m_{ir}) - m_{ir}\theta_i}{\sqrt{N_{1h}^\delta}} &= \frac{\sum_{\ell=1}^{\lfloor b_{ir} s \rfloor}(U_{i\ell} - \theta_i)}{\sqrt{b_{ir}}} \cdot \frac{\sqrt{b_{ir}}}{\sqrt{N_{1h}^\delta}} \\ &\Rightarrow u_i \mathcal{B}^{iu}(s) \cdot \sqrt{p_i}, \end{aligned}$$

where $\mathcal{B}^{iu}(s)$ is a standard Brownian motion process. Similarly, we obtain that

$$\frac{V_i(n_{ir}) - n_{ir}\theta_i}{\sqrt{N_{1h}^\delta}} \Rightarrow v_i \mathcal{B}^{iv}(s) \cdot \sqrt{q_i},$$

where $\mathcal{B}^{iv}(s)$ is also a standard Brownian motion process that is independent of $\mathcal{B}^{iu}(s)$ due to the assumption of independence between $\{U_{ij}, j = 1, \ldots, m_{ir}\}$ and $\{V_{i\ell}, \ell = 1, \ldots, n_{ir}\}$. Then, as $\delta \to 0$,

$$\begin{aligned} & \frac{U_1(m_{1r}) + V_1(n_{1r}) - U_h(m_{hr}) - V_h(n_{hr})}{\sqrt{N_{1h}^\delta}} \\ = \; & \frac{[U_1(m_{1r}) - m_{1r}\theta_1] + [V_1(n_{1r}) - n_{1r}\theta_1] - [U_h(m_{hr}) - m_{hr}\theta_j] - [V_h(n_{hr}) - n_{hr}\theta_h]}{\sqrt{N_{1h}^\delta}} + \frac{r(\theta_1 - \theta_h)}{\sqrt{N_{1h}^\delta}} \\ \Rightarrow \; & \sqrt{p_1} u_1 \mathcal{B}^{1u}(s) + \sqrt{q_1} v_1 \mathcal{B}^{1v}(s) + \sqrt{p_h} u_h \mathcal{B}^{hu}(s) + \sqrt{q_h} v_h \mathcal{B}^{hv}(s) + \sqrt{2a(p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2)} \cdot s, \end{aligned}$$

where the last part is due to the definition of N_{1h}^δ and $r = s N_{1h}^\delta$ and the assumption that $\theta_1 - \theta_h = \delta$.

By Theorem 11.4.5 in Whitt [45], we know that

$$\begin{aligned}\left\{\widetilde{Z}_{1h}(s) : 0 < s \le 1\right\} \;\Rightarrow\; &\left\{ \frac{1}{\sqrt{(p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2)}} \Big[\sqrt{p_1} u_1 \mathcal{B}^{1u}(s) + \sqrt{q_1} v_1 \mathcal{B}^{1v}(s) \right. \\ &\left. + \sqrt{p_g} u_h \mathcal{B}^{hu}(s) + \sqrt{q_h} v_h \mathcal{B}^{hv}(s) \Big] + \sqrt{2a} \cdot s : 0 < s \le 1 \right\} \\ \stackrel{\mathcal{D}}{=} \;&\left\{ \mathcal{B}(s) + \Delta s : 0 < s \le 1 \right\}, \end{aligned} \tag{29}$$

where the last equation is due to the independence among $\mathcal{B}^{1u}(s)$, $\mathcal{B}^{1v}(s)$, $\mathcal{B}^{hu}(s)$ and $\mathcal{B}^{hv}(s)$, and the notation $\stackrel{\mathcal{D}}{=}$ means "equal in distribution", and $\Delta = \sqrt{2a}$. Combined the results in Equations (28) and (29), we have

$$\left\{\widetilde{Z}_{1h}(s) : 0 \le s \le 1\right\} \;\Rightarrow\; \left\{\mathcal{B}(s) + \Delta s : 0 \le s \le 1\right\},$$

which concludes the proof. □

From the proof of Lemma 1, it is interesting to point out that the construction of the stochastic process $\widetilde{Z}_{1h}(s)$ is to scale the original process $Z_{1h}(r)$ by the term $\frac{1}{\sqrt{p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2}} \cdot \frac{r}{\sqrt{N_{1h}^{\delta}}}$ in order to establish the limiting process $\mathcal{B}_{\Delta}(s)$. Recall that, in Procedure 1, the elimination decision of system i is made at stage r when there exists some surviving system h such that the following condition is satisfied, that is,

$$Z_{ih}(r) < \min\left\{0, -\frac{a\hat{\sigma}_{ih}^2(r)}{\delta r} + \frac{\delta}{2}\right\}. \tag{30}$$

In order to make the comparison condition also hold for $\widetilde{Z}_{1h}(s)$, we need to scale the RHS in Inequality (30) by the same term. For simplicity of presentation, we define the upper bound and lower bound as follows,

$$\Gamma_{ih}(r) = \max\left\{0, \frac{a\hat{\sigma}_{1h}^2(r)}{\delta r} - \frac{\delta}{2}\right\} \text{ and } -\Gamma_{ih}(r) = \min\left\{0, -\frac{a\hat{\sigma}_{1h}^2(r)}{\delta r} + \frac{\delta}{2}\right\},$$

which forms the symmetric continuation region Λ_{ih} for the pair of systems i and h. Since we are interested in the pair of systems 1 and h, where $h = 2, 3, \ldots, k$, we next consider only the continuation region Λ_{1h}, for $h = 2, 3, \ldots, k$.

Let $\Gamma_{1h}^{\delta}(s)$ be the upper bound for $\widetilde{Z}_{1h}(s)$, which is defined as

$$\begin{aligned}\Gamma_{1h}^{\delta}(s) &= \frac{1}{\sqrt{p_1u_1^2+q_1v_1^2+p_hu_h^2+q_hv_h^2}}\cdot\frac{r}{\sqrt{N_{1h}^{\delta}}}\cdot\Gamma_{ih}(r) \qquad (31)\\ &= \max\left\{0,\frac{a\hat{\sigma}_{1h}^2(r)}{\delta\sqrt{p_1u_1^2+q_1v_1^2+p_hu_h^2+q_hv_h^2}\sqrt{N_{1h}^{\delta}}}-\frac{\delta r}{2\sqrt{p_1u_1^2+q_1v_1^2+p_hu_h^2+q_hv_h^2}\sqrt{N_{1h}^{\delta}}}\right\}.\end{aligned}$$

Then, the upper boundary $\Gamma_{1h}^{\delta}(s)$ and lower boundary $-\Gamma_{1h}^{\delta}(s)$ forms the symmetric continuation region Λ_{1h}^{δ} for $\widetilde{Z}_{1h}(\cdot)$, implying that either system 1 or h is eliminated depends on whether $\widetilde{Z}_{1h}(\cdot)$ exits the continuation region Λ_{1h}^{δ} from above or below. We are now ready to establish the second important result.

Lemma 2 (Convergence of the Continuation Region)**.** *Define the symmetric continuation region Λ_{1h}^{δ} for $\widetilde{Z}_{1h}(\cdot)$ by the upper boundary $\Gamma_{1h}^{\delta}(s)$ and lower boundary $-\Gamma_{1h}^{\delta}(s)$ as in Equation (31). Suppose that the conditions in Theorem 1 are all satisfied. Then, under the SC, as $\delta\to 0$, we have*

$$\Gamma_{1h}^{\delta}(s)\to\Gamma(s)=\max\left\{0,\frac{a}{\Delta}-\frac{\Delta}{2}\cdot s\right\},\quad w.p.1,$$

where $\Delta=\sqrt{2a}$. Moreover, the asymptotic region Λ, formed by $\Gamma(s)$ and $-\Gamma(s)$, is a symmetric triangular region for the Brownian motion process $\mathcal{B}_{\Delta}(\cdot)$ as obtained in Lemma 1.

Proof. We start by analyzing the variance estimator $\hat{\sigma}_{1h}^2(r)$ in Equation (5), which is rewritten as follows:

$$\hat{\sigma}_{1h}^2(r) = \frac{1}{r}\left[m_{1r}S_{\mathrm{CV}}^2(1,r)+n_{1r}S_{\mathrm{CV+CE}}^2(1,r)+m_{hr}S_{\mathrm{CV}}^2(h,r)+n_{hr}S_{\mathrm{CV+CE}}^2(h,r)\right].$$

Recall that $m_{ir}/r\to p_i$ and $n_{ir}/r\to q_i=1-p_i$, $i=1$ or h, w.p.1 as assumed in Theorem 1. We first consider the case when both p_i and q_i are strictly positive. As $\delta\to 0$, $n_0\to\infty$, so that $N_{1h}^{\delta}\to\infty$ and $r\to\infty$, implying that $S_{\mathrm{CV}}^2(i,r)\to u_i^2$ and $S_{\mathrm{CV+CE}}^2(i,r)\to v_i^2$, w.p.1. Therefore, as $\delta\to 0$, we know that

$$\hat{\sigma}_{1h}^2(r)\to p_1u_1^2+q_1v_1^2+p_hu_h^2+q_hv_h^2,\quad \text{w.p.1}, \qquad (32)$$

for any positive number $p_i>0$ and $q_i>0$, where $i=1$ or h.

For the case that either $p_i=0$ or $q_i=0$, the conclusion in (32) also holds. For instance, if $p_i=0$, then $S_{\mathrm{CV}}^2(i,r)$ is bounded or $S_{\mathrm{CV}}^2(i,r)\to u_i^2$ w.p.1, depending on whether m_{ir} is finite or $m_{ir}\to\infty$ as $r\to\infty$. However, no matter how $S_{\mathrm{CV}}^2(i,r)$ evolves, the term $\frac{m_{ir}}{r}S_{\mathrm{CV}}^2(i,r)\to 0$ w.p.1 as $r\to\infty$ since that $m_{ir}/r\to 0$. If $p_i=1$, i.e., $q_i=0$, similar argument can be applied to the term

$\frac{n_{ir}}{r} S^2_{\text{CV+CE}}(i,r)$, which converges to 0 w.p.1. Therefore, for any $p_i \in [0,1]$, $i = 1$ or h, we always have the result in (32). Recall the definition of $N^{\delta}_{1h} = \lceil 2a(p_1 u_1^2 + q_1 v_1^2 + p_h u_h^2 + q_h v_h^2)/\delta^2 \rceil$, then we obtain that

$$\Gamma^{\delta}_{1h}(s) \to \Gamma(s) = \max\left\{0, \frac{a}{\Delta} - \frac{\Delta}{2} \cdot s\right\}, \quad \text{w.p.1 as } \delta \to 0,$$

which concludes the proof. □

Lemma 1 demonstrates the weak convergence of $\widetilde{Z}_{1h}(\cdot)$ to $\mathcal{B}_{\Delta}(\cdot)$ on $[0,1]$, and Lemma 2 establishes the convergence of the corresponding continuation region Λ^{δ}_{1h} to the triangular region Λ. However, elimination decisions are only made at these stopping times when the stochastic processes first exit the region.

Let T^{δ}_{1h} denote the stopping time at which $\widetilde{Z}_{1h}(\cdot)$ first exits the continuation region Λ^{δ}_{1h}, i.e.,

$$T^{\delta}_{1h} = \inf\left\{s : \left|\widetilde{Z}_{1h}(s)\right| \geq \Gamma^{\delta}_{1h}(s)\right\},$$

and let T_{1h} denote the stopping time at which $\mathcal{B}_{\Delta}(\cdot)$ first exits the triangular region Λ, i.e.,

$$T_{1h} = \inf\left\{s : |\mathcal{B}_{\Delta}(s)| \geq \Gamma(s)\right\}.$$

In order to bound the probability of incorrect selection, we need a stronger result to ensure that the value at the stopping time $\widetilde{Z}_{1h}(T^{\delta}_{1h})$ can be approximated by $\mathcal{B}_{\Delta}(T_{1h})$, which can be guaranteed by the following lemma.

Lemma 3 (Convergence at Stopping Times)**.** *Suppose that the conditions in Theorem 1 are all satisfied. Then, as* $\delta \to 0$*,*

$$\widetilde{Z}_{1h}(T^{\delta}_{1h}) \Rightarrow \mathcal{B}_{\Delta}(T_{1h}).$$

We omit the proof of Lemma 3 here, since it follows exactly the same logic as that of proving Proposition 3.2 of Kim et al. [24] and Lemma 2 in Luo et al. [29]. Based on the results in Lemmas 1 to 3, we are now ready to prove Theorem 1.

Proof. We first consider the SC where $\theta_1 - \delta = \theta_2 = \cdots = \theta_k$. Then, we can bound the PCS, i.e.,

the probability of selecting system 1, as follows,

$$\begin{aligned}\liminf_{\delta\to 0}\ \mathbb{P}\left\{\text{select system 1}\right\} &= \liminf_{\delta\to 0}\left[1-\mathbb{P}\left\{\bigcup_{h=1}^{k-1}\left\{\text{system } h \text{ eliminates 1}\right\}\right\}\right] \\ &\geq 1-\limsup_{\delta\to 0}\sum_{h=1}^{k-1}\mathbb{P}\left\{\text{system } h \text{ eliminates 1}\right\}, \qquad (33)\end{aligned}$$

where (33) is due to Bonferroni inequality (e.g., refer to Kim and Nelson [23]). Note that the probability of incorrect selection between systems 1 and h, i.e., the probability that system 1 is eliminated by system h, is

$$\begin{aligned}\limsup_{\delta\to 0}\ \mathbb{P}\left\{\text{system } h \text{ eliminates 1}\right\} &= \limsup_{\delta\to 0}\ \mathbb{P}\left\{\widetilde{Z}_{1h}\left(T_{1h}^{\delta}\right)\leq 0\right\} \qquad (34)\\ &= \mathbb{P}\left\{\mathcal{B}_{\Delta}\left(T_{1h}\right)\leq 0\right\} \qquad (35)\\ &= \frac{1}{2}e^{-\frac{a}{\Delta}\Delta}=\frac{\alpha}{k-1}, \qquad (36)\end{aligned}$$

where (34) denotes the probability that system h eliminates system 1 since $Z_{1h}(\cdot)$ exits the continuation region through the lower boundary, (35) follows from Lemma 3, and (36) follows from Fabian's result in Fabian [12]. Plugging (36) into (33) yields

$$\liminf_{\delta\to 0}\ \mathbb{P}\left\{\text{select system 1}\right\} \geq 1-\sum_{h=1}^{k-1}\frac{\alpha}{k-1}=1-\alpha.$$

For general cases under the IZ formulation, i.e., $\theta_1-\delta\geq\theta_2\geq\cdots\geq\theta_k$, the stochastic process $\widetilde{Z}_{1h}(\cdot)$ defined in (27) no longer converges in distribution to the Brownian motion process $\mathcal{B}_{\Delta}(\cdot)$. However, we can define

$$\widehat{Z}_{1h}\left(s\right)=\frac{1}{\sqrt{p_1u_1^2+q_1v_1^2+p_hu_h^2+q_hv_h^2}}\cdot\frac{U_1(m_{1r})+V_1(n_{1r})-U_h(m_{hr})-V_h(n_{hr})-(\theta_1-\theta_h-\delta)r}{\sqrt{N_{1h}^{\delta}}}.$$

By Lemma 1, we know that $\widehat{Z}_{1h}\left(\cdot\right)\Rightarrow\mathcal{B}_{\Delta}(\cdot)$ as $\delta\to 0$. Moreover,

$$\widehat{Z}_{1h}\left(\cdot\right)\leq\widetilde{Z}_{1h}\left(\cdot\right), \quad a.s. \qquad (37)$$

Define $\widehat{T}_{1h}^{\delta}$ as the stopping time at which $\widehat{Z}_{1h}(\cdot)$ first exits the continuation region Λ_{1h}^{δ}, i.e.,

$$\widehat{T}_{1h}^{\delta}=\inf\left\{s:\left|\widehat{Z}_{1h}\left(s\right)\right|\geq\Gamma_{1h}^{\delta}(s)\right\}.$$

Then, the probability of incorrect selection between systems 1 and h is

$$\begin{aligned}
\limsup_{\delta\to 0}\ \mathbb{P}\left\{\text{system } h \text{ eliminates } 1\right\} &= \limsup_{\delta\to 0}\ \mathbb{P}\left\{\widetilde{Z}_{1h}\left(T_{1h}^{\delta}\right)\leq 0\right\} \\
&\leq \limsup_{\delta\to 0}\ \mathbb{P}\left\{\widehat{Z}_{1h}\left(\widehat{T}_{1h}^{\delta}\right)\leq 0\right\} \qquad (38)\\
&= \mathbb{P}\left\{\mathcal{B}_{\Delta}\left(T_{1h}\right)\leq 0\right\} \\
&= \frac{\alpha}{k-1}, \qquad (39)
\end{aligned}$$

where (38) follows from (37). Plugging (39) into (33) concludes the proof of the theorem. □

B Fully Sequential Selection Procedures

B.1 TN-like Procedure with CV and CE

We present a TN-like procedure that can employ the CV+CE combined models described in Section 3.2. The variance estimator is computed in the first stage and then fixed in the subsequent stages. The finite-time statistical validity can be justified by using a similar argument as in Theorem 1 of Tsai and Nelson [43]. That is, we can guarantee that $\Pr\{\text{select system } 1|\theta_1 \geq \theta_2+\delta\} \geq 1-\alpha$ if any of the Combined Models I, II, III holds. In the following procedure description, we assume that Combined Model I holds and use its corresponding notation presented in Section 3.2.1.

TN-like Procedure with CV+CE

Step 0. Setup: Select confidence level $1/k < 1-\alpha < 1$, IZ parameter $\delta > 0$, the preliminary-stage sample size $m_0 > q+2$, and the first-stage sample size $n_0 \geq 2$. Let $a = 2\eta \times (n_0-1)$, where

$$\eta = \frac{1}{2}\left\{\left[2\left(1-(1-\alpha)^{\frac{1}{k-1}}\right)\right]^{-\frac{2}{n_0-1}}-1\right\}.$$

Step 1. Initialization: Let $I = \{1,2,\ldots,k\}$ be the set of systems still in contention. For each system $i \in I$, generate $\{(Y_{ij},\mathbf{C}_{ij},\mathbf{X}_{ij}),j=1,2,\ldots,m_0\}$ and then compute the estimator $\widehat{\boldsymbol{\beta}}_i^{(1)}(m_0)$ according to Equation (12). Let r be the observation counter. Set $r = n_0$. For each system $i \in I$, perform additional independent sampling to generate $\{(Y_{ij},\mathbf{C}_{ij},\mathbf{X}_{ij}),j = m_0+1, m_0+2,\ldots,m_0+n_0\}$, and then based on which, compute the controlled sample mean:

$$\widehat{\theta}_i(r) \equiv \bar{Y}_i^{(1)}\left[m_0, m_0+r\right] = \frac{1}{r}\sum_{j=m_0+1}^{m_0+r}\left[\mathrm{E}[Y_{ij}|\mathbf{X}_{ij}] - \left(\mathrm{E}[\mathbf{C}_{ij}|\mathbf{X}_{ij}] - \boldsymbol{\mu}_i\right)^T\widehat{\boldsymbol{\beta}}_i^{(1)}(m_0)\right].$$

For all $i \neq h$, calculate the sample variance of the observation difference between any two

systems, $S^2_{ih}(r)$, which is analogous to $S^2_{ih}[m_0, m_0 + r]$ (defined in Equation (9)) with the application of Combined Model I.

Step 2. Elimination:

Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \setminus \left\{ i \in I^{\text{old}} : \widehat{\theta}_i(r) - \widehat{\theta}_h(r) < \min\left\{0, -\frac{a}{2\delta} \cdot \frac{S^2_{ih}(n_0)}{r} + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i \right\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 3. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r + 1$ and take the rth sample $(Y_{i(m_0+r)}, \mathbf{C}_{i(m_0+r)}, \mathbf{X}_{i(m_0+r)})$ from system $i \in I$. Update $\widehat{\theta}_i(r)$ for each system $i \in I$ and go to Step 2.

B.2 KN-like Procedure with Variance Updating

KN-like Procedure with Variance Updating

Step 0. Setup: Select confidence level $1/k < 1 - \alpha < 1$, IZ parameter $\delta > 0$ and the first-stage sample size $n_0 \geq 2$. Let $a = -\log[2\alpha/(k-1)]$.

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. Obtain n_0 outputs $X_{ij}, j = 1, 2, \cdots, n_0$, from each system $i \in I$. Let r be the observation counter. Set $r = n_0$.

Step 2. Update: Calculate the sample mean of the first r outputs from system i

$$\bar{X}_i(r) = \frac{1}{r}\sum_{j=1}^{r} X_{ij}.$$

For all $i \neq h$, calculate the sample variance of the difference between systems i and h,

$$S^2_{ih}(r) = \frac{1}{r-1}\sum_{j=1}^{r}(X_{ij} - X_{hj} - [\bar{X}_i(r) - \bar{X}_h(r)])^2.$$

Step 3. Elimination: Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \setminus \left\{ i \in I^{\text{old}} : \bar{X}_i(r) - \bar{X}_h(r) < \min\left\{0, -\frac{a}{\delta} \cdot \frac{S^2_{ih}(r)}{r} + \frac{\delta}{2}\right\} \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i \right\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 4. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r + 1$ and take the rth sample X_{ir} from system $i \in I$, and go to Step 2.

B.3 TN-like Procedure with Variance Updating

TN-like Procedure with Variance Updating

Step 0. Setup: Select confidence level $1/k < 1-\alpha < 1$, IZ parameter $\delta > 0$, the preliminary-stage sample size $m_0 > q + 2$, and the first-stage sample size $n_0 \geq 2$. Let $a = -\log[2\alpha/(k-1)]$.

Step 1. Initialization: Let $I = \{1, 2, \ldots, k\}$ be the set of systems still in contention. For each system $i \in I$, generate $\{(Y_{ij}, \mathbf{C}_{ij}), j = 1, 2, \ldots, m_0\}$ and then compute the estimator $\widehat{\boldsymbol{\beta}}_i(m_0)$ according to Equation (7). For each system $i \in I$, perform additional independent sampling to generate $\{(Y_{ij}, \mathbf{C}_{ij}), j = m_0 + 1, m_0 + 2, \ldots, m_0 + n_0\}$. Let r be the observation counter. Set $r = n_0$.

Step 2. Update: Calculate the sample mean of the first r outputs (which starts after the preliminary stage) from system i

$$\widehat{\theta}_i(r) \equiv \bar{Y}_i[m_0, m_0 + r] = \frac{1}{r} \sum_{j=m_0+1}^{m_0+r} \left[Y_{ij} - (\mathbf{C}_{ij} - \boldsymbol{\mu}_i)^T \widehat{\boldsymbol{\beta}}_i(m_0) \right].$$

For all $i \neq h$, calculate the sample variance of the observation difference between any two systems, $S^2_{ih}(r) \equiv S^2_{ih}[m_0, m_0 + r]$ (defined in Equation (9)).

Step 3. Elimination: Set $I^{\text{old}} = I$. Let

$$I = I^{\text{old}} \setminus \left\{ i \in I^{\text{old}} : \widehat{\theta}_i(r) - \widehat{\theta}_h(r) < \min\left\{ 0, -\frac{a}{\delta} \cdot \frac{S^2_{ih}(r)}{r} + \frac{\delta}{2} \right\} \text{ for some } h \in I^{\text{old}} \text{ and } h \neq i \right\},$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Step 4. Stopping Rule: If $|I| = 1$, then stop and select the system whose index is in I as the best. Otherwise, let $r = r+1$ and take the rth sample $\left(Y_{i(m_0+r)}, \mathbf{C}_{i(m_0+r)}\right)$ from system $i \in I$, and go to Step 2.

C Numerical Results for CV+CE

We investigate the effect of different levels of squared correlations ($R^2_{Y,X}$ and $R^2_{X,C}$) on the efficiency of the $\mathcal{TN}$-like procedure presented in Appendix B.1 when using the Combined Models I, II, and III

and compare them to $\mathcal{KN}$ and $\mathcal{TN}$ under the slippage and MDM configurations (see Table 9 and 10). In Combined Model I, we set $\rho = 0.7$. Notice that for the MDM configuration fewer observations from each system are consumed when the number of systems increases since the additional systems are far from the best (i.e., more easily eliminated). It can be shown that the performance of $\mathcal{TN}$ is better than $\mathcal{KN}$ as long as $R^2_{Y,C}$ is greater than or equal to 0.2, although the difference might not be significant. We can see that the $\mathcal{TN}$-like procedure applying either Combined Model I or II is more efficient than $\mathcal{TN}$ even when the conditional variable is not very effective (i.e., $R^2_{Y,X}$ is large). For instance, when $R^2_{Y,C} = R^2_{X,C} = 0.2$ and either Combined Model I or II is used, the $\mathcal{TN}$-like procedure (with $R^2_{Y,X} = 0.8$) outperforms $\mathcal{TN}$ in terms of significant reductions in ANS in all configurations. In addition, under the combined models, the efficiency improvement when using a better conditional variable is more significant compared to that under Model 0 (with a better control variable). For example, as shown in Table 9, when $k = 100$, $R^2_{Y,X} = 0.8$, and Combined Model I is used, we obtain a 47% reduction in ANS (134 to 71) when $R^2_{X,C}$ is increased from 0.2 to 0.5, while the ANS reduction is around 36% (199 to 127) for procedure $\mathcal{TN}$ when $R^2_{Y,C}$ is increased from 0.2 to 0.5. Further, in some cases employing Combined Model I or II will achieve the best efficiency in terms of variance reduction (which cannot be obtained by $\mathcal{KN}$ or $\mathcal{TN}$); we see that the ANS of CV+CE-I and CV+CE-II is as low as 30 when $R^2_{Y,X} = 0.2$ and $R^2_{X,C} = 0.8$. It should be noticed that ANS = 30 means that the choice of the best system can be made right after the first-stage observations have been collected because $m_0 + n_0 = 30$. We also find that the $\mathcal{TN}$-like procedure employing either Combined Model I or II can deliver more superior performance when the value of $R^2_{Y,X}$ is decreased (or the value of $R^2_{X,C}$ is increased), which is consistent with the previous analytical results. For instance, in Table 9, it can be observed that when employing Combined Model II, $k = 100$, and $R^2_{X,C} = 0.2$, the ANS is 159 in the case of $R^2_{Y,X} = 0.8$ while the ANS is reduced to 49 when $R^2_{Y,X} = 0.2$. However, the performance of Combined Model III is equivalent to that of Model 0 when using the same values of $R^2_{Y,C}$ and $R^2_{Y,X}$, which implies that the benefit of variance reduction from Combined Model III is not guaranteed. In the illustrative example presented in Section 6, we even find that the use of Combined Model III results in a worse performance when compared to Model 0. Overall, Combined Model I reveals a better performance in terms of ANS than Combined Model II, although Combined Model I is more difficult to apply in practice.

Table 9: Performance measures for our CV+CE fully sequential procedures in comparison with $\mathcal{KN}$ and $\mathcal{TN}$ under the SC when $m_0 = 10$, $n_0 = 20$, $q = 1$ and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	$R^2_{Y,X}$	$R^2_{X,C}$	$k = 30$		$k = 50$		$k = 100$	
				PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$				0.96	180	0.97	196	0.96	205
$\mathcal{TN}$	0.2			0.96	174	0.96	189	0.96	199
	0.5			0.99	109	0.98	119	0.99	127
	0.8			0.96	52	0.98	54	0.98	59
CV+CE-I		0.8	0.2	0.99	116	0.98	121	0.99	134
		0.8	0.5	0.99	63	0.99	66	0.99	71
		0.5	0.5	0.98	42	0.98	44	1	46
		0.2	0.8	0.98	30	1	30	1	30
CV+CE-II		0.8	0.2	0.97	144	0.96	156	0.97	159
		0.8	0.5	1	92	0.96	99	0.99	103
		0.5	0.5	1	62	0.96	64	0.96	70
		0.2	0.2	0.95	46	0.98	48	0.96	49
		0.2	0.8	0.98	30	1	30	1	30
CV+CE-III		0.2		0.96	171	0.96	189	0.96	199
		0.5		0.99	109	0.98	119	0.99	127
		0.8		0.96	52	0.98	54	0.98	59

Table 10: Performance measures for our CV+CE fully sequential procedures in comparison with $\mathcal{KN}$ and $\mathcal{TN}$ under the MDM configuration when $m_0 = 10$, $n_0 = 20$, $q = 1$ and $1 - \alpha = 0.95$.

	$R^2_{Y,C}$	$R^2_{Y,X}$	$R^2_{X,C}$	$k = 30$		$k = 50$		$k = 100$	
				PCS	ANS	PCS	ANS	PCS	ANS
$\mathcal{KN}$				1	165	1	125	1	86
$\mathcal{TN}$	0.2			1	155	1	114	1	81
	0.5			1	96	1	74	1	57
	0.8			1	50	1	45	1	38
CV+CE-I		0.8	0.2	0.98	94	1	75	1	53
		0.5	0.5	1	43	0.99	38	0.99	36
		0.2	0.8	0.99	31	1	30	1	30
CV+CE-II		0.8	0.2	1	122	1	96	1	66
		0.5	0.5	1	58	1	49	1	42
		0.2	0.2	1	45	1	41	1	36
		0.2	0.8	1	31	1	31	1	31
CV+CE-III		0.2		1	148	1	110	1	81
		0.5		1	96	1	74	1	57
		0.8		1	50	1	45	1	38